NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4120 Calculus 4K**

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Permitted examination support material: (Code C): Approved simple calculator.

Other information:
Every answer must be justified; describe clearly how you have reached your answers.

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Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
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Date Signature

Problem 1 Find the function $y_2(t)$ $(t \geq 0)$ solving the initial value problem

$$\begin{cases} y_1'(t) - y_2(t) = 0, & y_1(0) = 1, \\ y_2'(t) + 2y_2(t) + y_1(t) = 5u(t-3), & y_2(0) = 0, \end{cases}$$

where $u(t)$ is the unit step or Heaviside function.

Hint: Laplace transform. Do not compute $y_1(t)$.

Problem 2 Let the function f be defined by $f(x) = 1 - x$ for $x \in [0, 1]$.

(i) Find the Fourier sin series of $f(x)$,

(ii) sketch the sum of this series on the interval $[-2, 2]$, and

(iii) determine the sum of the series at $x = 0$.

Problem 3

(i) State the Cauchy-Riemann equations,

(ii) show that the function $g(z) = |z|^2$ is not analytic in any point $z \in \mathbb{C}$, and

(iii) find a conjugate harmonic function of $u(x, y) = 5xy - 1$.

Problem 4 Let C be the circle $|z| = 2$ with counter clockwise orientation. Calculate the following integrals:

$$(i)\ \oint_C \sin z\, dz, \quad (ii)\ \oint_C \left(\frac{3}{z-i} + \frac{2}{(z-i)^2} + \frac{1}{(z-i)^3} \right) dz, \quad (iii)\ \oint_C |z|^2\, dz.$$

Problem 5 Calculate the integral

$$\int_0^{2\pi} \frac{1}{4 - 2\cos\theta}\, d\theta.$$

Hint: Residue integration.

Problem 6 Find a Laurent series centered at $z_0 = 0$ of

$$h(z) = \operatorname{Ln} \frac{z^2 - 1}{2z^2}.$$

Can $h(z)$ be represented by Laurent series centered at $z_0 = 0$ in other parts of the complex plane? Explain.

Problem 7 A rectangular metal plate in $R = [0,1] \times [0,1]$ is insulated on three sides and heated on the top. The stationary temperature distribution $u(x, y)$ is modelled by the boundary value problem

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0, & 0 < x < 1, \quad 0 < y < 1, \\ u_x(0,y) = 0 = u_x(1,y), & 0 < y < 1, \\ u_y(x,0) = 0, & 0 < x < 1, \\ u(x,1) = 2021 - 2020\cos(5\pi x), & 0 < x < 1. \end{cases}$$

Find the temperature $u(x, y)$.

Hint: You may use without justification that the problem

$$X''(x) - kX(x) = 0 \quad \text{in} \quad (0,1), \qquad X'(0) = 0 = X'(1),$$

has solutions $X \neq 0$ only when $k = -(n\pi)^2$ for $n = 0, 1, 2, 3, \ldots$, and that these solutions are given by

$$X_n(x) = A_n \cos(n\pi x) \qquad \text{for any} \qquad A_n \in \mathbb{R} \quad \text{and} \quad n = 0, 1, 2, 3, \ldots$$

Problem 8 Use the Fourier transform to find a solution $v(x, t)$ of

$$\begin{cases} v_t(x,t) - 2v_x - 5v_{xx}(x,t) = 0, & x \in \mathbb{R}, \quad t > 0, \\ v(x,0) = \delta(x), & x \in \mathbb{R}, \end{cases}$$

where $\delta(x)$ is the delta function.

Hint: You may assume the Fourier transforms in the x-variable of v and its derivatives exist and that $\lim_{x \to \pm\infty} v(x,t) = 0 = \lim_{x \to \pm\infty} v_x(x,t)$.

Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero except at $t = a$, $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, and $\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$ for any continuous function g.
- **Convolution**

  For functions defined on the real line:
  $$f * g\,(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}.$$
  For functions defined only on the positive half-axis:
  $$f * g\,(x) = \int_{0}^{x} f(y)g(x-y)dy, \quad x > 0.$$

Laplace transform

- Definition: $\mathcal{L}[f](s) = F(s) = \int_0^\infty f(t)e^{-st}dt$

General formulas		$f(t)$	$F(s)$
		1	$\frac{1}{s}$
$\mathcal{L}[e^{at}f(t)](s) = F(s-a)$		$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
$\mathcal{L}[f'](s) = s\mathcal{L}[f] - f(0)$		e^{at}	$\frac{1}{s-a}$
$\mathcal{L}[f''](s) = s^2\mathcal{L}[f] - sf(0) - f'(0)$		$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right](s) = \frac{1}{s}\mathcal{L}[f]$		$\cos bt$	$\frac{s}{s^2+b^2}$
$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$		$\sin bt$	$\frac{b}{s^2+b^2}$
$\mathcal{L}[f(t-c)u(t-c)](s) = e^{-cs}F(s),\ c > 0$		$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{L}[tf(t)](s) = -F'(s)$		$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty F(\sigma)d\sigma$		$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
		$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- $2L$-periodic functions, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx$$

- Functions defined on the whole real line (need not be periodic)

$$\hat{f}(w) = \mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx,$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw.$$

- Parseval's identities

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2, \qquad \int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|\hat{f}(w)|^2dw$$

General formulas		$f(x)$	$\hat{f}(w)$
$\widehat{f'(x)} = iw\hat{f}(w)$		$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-iaw}$
$\widehat{f''(x)} = -w^2\hat{f}(w)$		$\begin{cases}1, & -b \le x \le b \\ 0, & \lvert x\rvert > b\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
$f\widehat{(x-a)} = e^{-iaw}\hat{f}(w)$		$e^{-ax}u(x)$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\hat{f}(w-b) = e^{\widehat{ibx}f(x)}$		$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$
$\widehat{f*g} = \sqrt{2\pi}\hat{f}\hat{g}$		e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i \sin y)$
- $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}, \qquad \cosh z = \frac{e^{z}+e^{-z}}{2}, \quad \sinh z = \frac{e^{z}-e^{-z}}{2}$
- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad b_n = \frac{1}{2\pi i} \oint_C f(z)(z-z_0)^{n-1} dz$$

Some useful integrals

$\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$

$\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$

$\int x^2 \sin ax \, dx = \frac{2}{a^2} x \sin ax + \frac{2-a^2x^2}{a^3} \cos ax + C$

$\int x^2 \cos ax \, dx = \frac{2}{a^2} x \cos ax - \frac{2-a^2x^2}{a^3} \sin ax + C$

$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$

$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$

$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \ a > 0$

Some trigonometric identities

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

Some important series

- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, $\quad \sum_{n=0}^{\infty} x^n$ diverges for $|x| \geq 1$.
- $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for $x \in \mathbb{R}$.