Problem 1 Laplace transforming both equations and using the initial conditions gives

$$\begin{cases} sY_1 - y_1(0) - Y_2 = 0 \\ sY_2 - y_2(0) + 2Y_2 + Y_1 = 5\frac{1}{s}e^{-3s} \end{cases} \implies \begin{cases} sY_1 - 1 - Y_2 = 0 \\ sY_2 + 2Y_2 + Y_1 = \frac{5}{s}e^{-3s}. \end{cases}$$

We then solve the system of equations for Y_2:

$$\begin{cases} Y_1 = \frac{1}{s}(Y_2 + 1) \\ (s+2)Y_2 + \frac{1}{s}(Y_2+1) = \frac{5}{s}e^{-3s} \end{cases} \implies \frac{s^2+2s+1}{s}Y_2 = \frac{5e^{-3s}-1}{s}$$

$$\implies \quad Y_2 = \frac{5e^{-3s}-1}{s^2+2s+1} = \frac{5e^{-3s}-1}{(s+1)^2}$$

Taking the inverse Laplace transform and using the t-shift formula, we find that

$$y_2 = \mathcal{L}^{-1}[Y_2] = 5g(t-3)u(t-3) - g(t),$$

where by the s-shift formula and the tables,

$$g(t) = \mathcal{L}^{-1}\left[\tfrac{1}{(s+1)^2}\right](t) = e^{-t}\mathcal{L}^{-1}\left[\tfrac{1}{s^2}\right] = e^{-t}t.$$

Hence

$$\underline{y_2(t) = 5e^{-(t-3)}(t-3)u(t-3) - e^{-t}t}$$

Problem 2

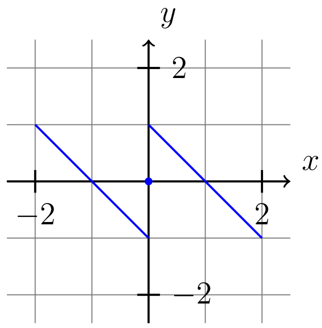
(i) The Fourier sin series of f is the Fourier series of the odd 2-periodic extension g of f. Since g is an odd extension of f, it follows that $a_0 = a_n = 0$, and

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \overset{L=1}{=} 2\int_0^1 (1-x)\sin n\pi x\,dx \\ &\overset{i.b.p.}{=} \left[2(1-x)\frac{-1}{n\pi}\cos n\pi x\right]_0^1 - \int_0^1 (-2)\frac{-1}{n\pi}\cos n\pi x\,dx \\ &= \frac{-2}{n\pi}(0-1) - \frac{2}{n\pi}\left[\frac{1}{n\pi}\sin n\pi x\right]_0^1 = \frac{2}{n\pi} + 0. \end{aligned}$$

Thus we have the following Fourier sine series of f:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin n\pi x.$$

(ii) The sum S_g of the Fourier series corresponds to the odd 2-periodic extension g of f:



(iii) Since the extension g of f is piecewise continous and one-sided differentiable, we have

$$S_g(0) = \frac{g(0^+) + g(0^-)}{2} = \frac{1-1}{2} = \underline{0}.$$

Alternative solution: Since the partial sums $S_{g,k}(0) = \sum_{n=1}^{k} \frac{2}{n\pi} \sin n\pi 0 = 0$ for all k, it follows that $S_g(0) = \lim_{k\to\infty} S_{g,k}(0) = \lim_{k\to\infty} 0 = 0$

Problem 3

(i) $u_x = v_y$, $u_y = -v_x$

(ii) We have $g(z) = |z|^2 \overset{z=x+iy}{=} x^2 + y^2 = u + iv$, and thus we identify $u = x^2 + y^2$ and $v = 0$. We then check the Cauchy Riemann equations:

$$\begin{aligned} u_x &= 2x \neq 0 = v_y, && \text{for } x \neq 0, \\ u_y &= 2y \neq 0 = -v_x, && \text{for } y \neq 0, \end{aligned}$$

Thus the Cauchy-Riemann equations do not hold for $z \neq 0$, which implies that g is not analytic in $\mathbb{C}\setminus\{0\}$. The function is then also not analytic at $z = 0$, because by definition, to be analytic at point it must be analytic in neighbourhood of the point.

(iii) Let v be the conjugate harmonic function of u. By the Cauchy-Riemann equations,

$$\begin{aligned} &(1)\ v_x = -u_y = -5x \\ &(2)\ v_y = u_x = 5y \end{aligned}$$

Solving by integrating (1), differentiating the result, and using (2), we find that

$$v \overset{(1)}{=} -\frac{5}{2}x^2 + C(y) \quad \Longrightarrow \quad v_y = 0 + C'(y) \overset{(2)}{=} 5y \quad \Longrightarrow \quad C(y) = \frac{5}{2}y^2 + C,$$

where $C \in \mathbb{R}$ is a constant of integration. The result is then

$$v(x,y) = -\frac{5}{2}(x^2 - y^2) + C.$$

Problem 4

(i) Since $\sin z$ is analytic in $\mathbb{C}$, we can use Cauchy's integral theorem to conclude that

$$\oint_C \sin z dz = 0.$$

(ii) Let $f(z) = \frac{3}{z-i} + \frac{2}{(z-i)^2} + \frac{1}{(z-i)^3}$. Note that $f(z)$ only has one singularity within C, $z = i$, and that $f(z)$ is its own Laurent series about $z = i$. This (finite) series is defined ("converges") for all $|z| > 0$, and the coefficient of the $\frac{1}{z-i}$-term is then $\operatorname{Res}_{z=i} f(z) = 3$. By the Residue theorem we then have

$$\oint_C \left(\frac{3}{z-i} + \frac{2}{(z-i)^2} + \frac{1}{(z-i)^3}\right)dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = 6\pi i.$$

(iii) $f(z) = |z|^2$ is not analytic. We parametrise C: $z(t) = 2e^{it}, t \in [0, 2\pi)$, and compute

$$\oint_C |z|^2 dz = \int_0^{2\pi} |z(t)|^2 z'(t)dt = \int_0^{2\pi} |z(t)|^2 z'(t)dt = \int_0^{2\pi} 2^2 \cdot 2ie^{it}dt = 0$$

since e^{it} is 2π-periodic.

Problem 5 Trick: Use $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z(\theta) + \frac{1}{z(\theta)})$, where $z(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi)$, is a parametrisation of $|z| = 1$. Thus $dz = z'(\theta)d\theta = ie^{i\theta}d\theta = iz\, d\theta$, and

$$\begin{aligned} I &= \int_0^{2\pi} \frac{1}{4 - 2\cos\theta}\, d\theta = \int_0^{2\pi} \frac{1}{4 - (z(\theta) + \frac{1}{z(\theta)})}\, d\theta \\ &= \oint_{|z|=1} \frac{1}{4 - (z + \frac{1}{z})} \frac{dz}{iz} = i \oint_{|z|=1} \frac{1}{z^2 - 4z + 1} dz. \end{aligned}$$

Note that $z^2 - 4z + 1 = 0 \implies z = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3} = a_\pm$, and that $|a_-| < 1$ and $|a_+| > 1$. Therefore f has simple poles at $z = a_\pm$, and C encircles $z = a_-$ but not $z = a_+$. By the residue theorem and the formula for residue of simple pole:

$$I = i \cdot 2\pi i \operatorname*{Res}_{z=a_-} \left(\frac{1}{z^2 - 4z + 1}\right) = -2\pi \lim_{z \to a_-} \frac{(z - a_-)}{(z - a_-)(z - a_+)} = -2\pi \frac{1}{a_- - a_+} = \underline{\frac{\pi}{\sqrt{3}}}$$

Problem 6 To find the series, the idea is to use substitution, differentiate, use a geometric series, and finally, use termwise integration:

$$
\begin{aligned}
h(z) &= \mathrm{Ln}\frac{1}{2}(1-\frac{1}{z^2}) \overset{u=\frac{1}{z^2}}{=} \mathrm{Ln}\frac{1}{2}(1-u) = g(u) \\
\implies g'(u) &= \frac{1}{\frac{1}{2}(1-u)}(-\frac{1}{2}) = -\frac{1}{1-u} \overset{|u|<1}{=} -\sum_{n=0}^{\infty} u^n \\
\implies g(u) - g(0) &= \int_0^u g'(w)dw = \int_0^u \left(-\sum_{n=0}^{\infty} w^n\right) dw = -\sum_{n=0}^{\infty} \frac{u^{n+1}}{n+1} \\
\implies g(u) &= \mathrm{Ln}\,\frac{1}{2} - \sum_{k=1}^{\infty} \frac{u^k}{k}
\end{aligned}
$$

Thus, since $\mathrm{Ln}\,\frac{1}{2} = \mathrm{Ln}\,1 - \mathrm{Ln}\,2 = -\mathrm{Ln}\,2$,

$$h(z) = g(\frac{1}{z^2}) = -\mathrm{Ln}\,2 - \sum_{k=1}^{\infty} \frac{1}{kz^{2k}}.$$

By convergence of the geometric series, and the result that an term-wise integrated power series has the same radius of convergence as the original series, this series converges for $|u| < 1 \iff |z| > 1$.

Let $z = r \in \mathbb{R}$. Then $w = \frac{1}{2}(1-\frac{1}{z^2}) = \frac{1}{2}(1-\frac{1}{r^2}) = a_r \in \mathbb{R}$, and $a_r < 0$ for $|r| < 1$. Since $\mathrm{Ln}(w)$ is not analytic/has a singularity at $w = a_r$ for $|r| \leq 1$, by Laurent's (or Taylor's) theorem no Laurent (or Taylor) series of h can converge in an annulus (or disk) about $z = 0$ containing $z = r$ for $|r| \leq 1$. Hence h cannot be represented by a Laurent (or Taylor) series for $|z| < 1$, and the above series is the only Laurent series about $z = 0$.

Remark: If you try to expand in a series for $|z| < 1$, termwise integration will give a term like $\mathrm{Ln}\,z$, and this function does not have any power series representation for $|z| < 1$.

Problem 7 We proceed by separation of variables.

1) Find all solutions $u(x,y) = X(x)Y(y)$ solving the PDE and the homogenous (0) boundary conditions.

   (i) Derive the equations and boundary conditions for X and Y by inserting $u = XY$ into the PDE and boundary conditions:

$$
\begin{aligned}
X''Y + XY'' &= 0, && x, y \in (0,1) \\
X'(0)Y(y) &= 0 = X'(1)Y(y), && y \in (0,1) \\
X(x)Y'(0) &= 0, && x \in (0,1)
\end{aligned}
$$

Dividing by $XY(\neq 0)$ in the first equation, we get

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{konst} = k \quad \Longrightarrow \quad X'' - kX = 0, \quad Y'' + kY = 0.$$

From the next 3 equations, we get

$$\begin{array}{lll} X'(0) = 0 = X'(1) & \text{or} & Y = 0 \ (\implies u = 0), \\ Y'(0) = 0 & \text{or} & X = 0 \ (\implies u = 0). \end{array}$$

So either $u = 0$ or

(1) $$X'' - kX = 0, \quad X'(0) = 0 = X'(1),$$
(2) $$Y'' + kY = 0, \quad Y'(0) = 0.$$

(ii) Solve (1) for X: Using the hint, $X \neq 0$ only when $k = -(n\pi)^2$, and then

$$X(x) = X_n(x) = A_n \cos n\pi x \qquad \text{for} \qquad A_n \in \mathbb{R}, \quad n = 0, 1, 2, ...$$

(iii) Solve (2) for Y when $X \neq 0$: The characteristic equation $r^2 - (n\pi)^2 = 0$ has real solutions $r = \pm n\pi$, so the general solution is an exponential solution:

$$Y'' - (n\pi)^2 Y = 0 \quad \Longrightarrow \quad Y = \tilde{A}e^{n\pi y} + \tilde{B}e^{-n\pi y},$$

and the boundary condition then implies that

$$0 = Y'(0) = n\pi \tilde{A}e^0 - n\pi \tilde{B}e^0 \implies \tilde{A} = \tilde{B}.$$

We conclude that

$$Y(y) = Y_n(y) = 2\tilde{A}_n \cosh n\pi y \qquad \text{for} \qquad \tilde{A}_n \in \mathbb{R}, \quad n = 0, 1, 2, ...$$

(iv) All solutions $u = XY$ of the PDE and homogeneous boundary conditions are then on the form

$$u_n(x, y) = X_n(x)Y_n(y) = A_n \cosh(n\pi y) \cos(n\pi x) \qquad \text{for} \qquad A_n \in \mathbb{R}, \quad n = 0, 1, 2, ...$$

2) Solve the inhomogeneous boundary condition by superposition (linear combinations of solutions of linear homogeneous equations and conditions, solve the same equations and conditions). We therefore take the following candidate solution $u(x, y) = \sum_{n=0}^{\infty} u_n(x, y)$:

$$2021 - 2020 \cos 5\pi x = u(x, 1) = \sum_{n=0}^{\infty} A_n \cosh(n\pi) \cos(n\pi x)$$
$$\Longrightarrow \quad A_0 = 2021, \quad A_5 \cosh 5\pi = -2020, \quad a_n = 0 \text{ for } n \neq 0.$$

This gives the solution

$$\underline{u(x,y) = 2021 - 2020\frac{\cosh(5\pi y)}{\cosh(5\pi)}\cos(5\pi x)}$$

Note: In this case $u = u_0 + u_5$ solves the problem when A_0 and A_5 are defined as above, and we did not need to define u as an infinite sum.

Problem 8 Fourier transform (in x) the equation and initial values:

$$\begin{cases} \hat{v}_t - 5(iw)^2\hat{v} - 2(iw)\hat{v} = 0, & w \in \mathbb{R}, \quad t > 0, \\ \hat{v}(w,0) = \frac{1}{\sqrt{2\pi}}, & w \in \mathbb{R}, \end{cases}$$

Solve as an ODE in the variable t:

$$\hat{u}(w,t) = \frac{1}{\sqrt{2\pi}}e^{2iwt-5w^2t} = e^{2iwt}\left(\frac{1}{\sqrt{2\pi}}e^{-w^2t}\right) = e^{2iwt}\hat{g}(w,t).$$

Apply the inverse Fourier transform:

$$u(x,t) = \mathcal{F}^{-1}[\hat{u}(w,t)](x) = g(x+2t),$$

and

$$g(x,t) = \mathcal{F}^{-1}[\hat{g}(w,t)](x) \underset{a=\frac{1}{20t}}{\overset{\text{table}}{=}} \frac{1}{\sqrt{2\pi}}\sqrt{2a}e^{-ax^2} = \frac{1}{\sqrt{20\pi t}}e^{-\frac{x^2}{20t}}.$$

Hence

$$\underline{u(x,t) = \frac{1}{\sqrt{20\pi t}}e^{-\frac{(x+2t)^2}{20t}}.}$$