

Summary Lecture 23: Singularities

❶ **Singularity:** Point z_0 where $f(z)$ is not analytic/defined (...)

❷ **Isolated singularity:**

z_0 is the only singularity in $|z - z_0| < R$ for some $R > 0$

⇓ **Laurent's theorem**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\neq 0}, \quad 0 < |z - z_0| < R$$

Principal value of z_0 :

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

z_0 order n pole:

$$b_n \neq 0, b_k = 0, k > n$$

z_0 isolated essential singularity:

$$b_n \neq 0 \text{ for } \infty \text{ many } n$$

Summary Lecture 23: Residue integration

3 Residue integration:

$$\text{Goal: } \oint_C f(z) dz = 2\pi i \sum \text{residues}$$

4 Only one singularity:

z_0 only singularity of $f(z)$ enclosed by C (simple, closed, counter cl.wise)

↓ Laurent's theorem

$$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots, \quad 0 < |z - z_0| < R$$

↓ Term wise integration/Laurent's theorem

$$\oint_C f(z) dz = 2\pi i b_1$$

5 Residue of $f(z)$ at z_0 :

$$\operatorname{Res}_{z=z_0} f(z) = b_1$$

b_1 -coefficient in Laurent series that converges in $0 < |z - z_0| < R$!

Lecture 24: Residue integration

Kreyszig: Sections 16.3, 16.4

- 1 Residue integration
- 2 Formulas for residue of a pole
- 3 Residue theorem
- 4 Application: Computing real integrals
- 5 Examples

Frist øving 12 og 13:

Neste uke - onsdag kl 23:59!

Info om eksamen, øvinger, treffetid:

Se [www](http://www.ntnu.no).

Lecture 24: Residues

$\operatorname{Res}_{z=z_0} f(z) = b_1$ where

$$f(z) = \sum a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots, \quad 0 < |z - z_0| < R$$

z_0 order 1 pole:

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z) \quad \text{or} \quad \operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

z_0 order n pole:

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \left((z - z_0)^n f(z) \right)^{(n-1)}$$

z_0 isolated essential singularity:

Find b_1 from the Laurent series!

Lecture 24: The Residue theorem

Assume

(A1) C simple, closed curve oriented counterclockwise

(A2) $f(z)$ has singularities $z_1, \dots, z_m$ enclosed by C

Then

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^m \operatorname{Res}_{z=z_j} f(z).$$

Summary Lecture 24: Residue integration

- ① **Residues:** $\operatorname{Res}_{z=z_0} f(z) = b_1$ where

$$f(z) = \sum a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots, \quad 0 < |z-z_0| < R$$

z_0 order 1 pole:

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z-z_0)f(z)$$

or

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

z_0 order n pole:

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \left((z-z_0)^n f(z) \right)^{(n-1)}$$

z_0 isolated essential singularity:

Find b_1 from the Laurent series!

- ② **Residue theorem:** Assume

(A1) C simple, closed curve oriented counterclockwisely

(A2) $f(z)$ has finite number of singularities $z_1, \dots, z_m$ enclosed by C

Then

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^m \operatorname{Res}_{z=z_j} f(z)$$