

The simple proof of this important theorem is quite similar to that of Theorem 1 in Sec. 2.1 and is left to the student.

Verification of solutions in Probs. 2–13 proceeds as for ODEs. Problems 16–23 concern PDEs solvable like ODEs. To help the student with them, we consider two typical examples.

EXAMPLE 2 Solving $u_{xx} - u = 0$ Like an ODE

Find solutions u of the PDE $u_{xx} - u = 0$ depending on x and y .

Solution. Since no y -derivatives occur, we can solve this PDE like $u'' - u = 0$. In Sec. 2.2 we would have obtained $u = Ae^{x^2} + Be^{-x^2}$ with constant A and B . Here A and B may be functions of y , so that the answer is

$$u(x, y) = A(y)e^{x^2} + B(y)e^{-x^2}$$

with arbitrary functions A and B . We thus have a great variety of solutions. Check the result by differentiation. ■

EXAMPLE 3 Solving $u_{xy} = -u$, Like an ODE

Find solutions $u = u(x, y)$ of this PDE.

Solution. Setting $u_x = p$, we have $p_y = -p$, $p_y/p = -1$, $\ln |p| = -y + \tilde{c}(x)$, $p = c(x)e^{-y}$ and by integration with respect to x ,

$$u(x, y) = f(x)e^{-y} + g(y) \quad \text{where} \quad f(x) = \int c(x) dx,$$

here, $f(x)$ and $g(y)$ are arbitrary. ■

PROBLEM SET 12.1

Fundamental theorem. Prove it for second-order PDEs in two and three independent variables. *Hint.* Prove it by substitution.

13 VERIFICATION OF SOLUTIONS

Verify (by substitution) that the given function is a solution of the PDE. Sketch or graph the solution as a surface in space.

5. **Wave Equation (1)** with suitable c

- $u = x^2 + t^2$
- $u = \cos 4t \sin 2x$
- $u = \sin kct \cos kx$
- $u = \sin at \sin bx$

9. **Heat Equation (2)** with suitable c

- $u = e^{-t} \sin x$
- $u = e^{-\omega^2 t^2} \cos \omega x$
- $u = e^{-9t} \sin \omega x$
- $u = e^{-\pi^2 t} \cos 25x$

10–13 **Laplace Equation (3)**

- $u = e^{x^2} \cos y, e^{x^2} \sin y$
- $u = \arctan (y/x)$
- $u = \cos y \sinh x, \sin y \cosh x$

15. **Boundary value problem.** Verify that the function $u(x, y) = a \ln(x^2 + y^2) + b$ satisfies Laplace's equation

- (3) and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

16–23 PDEs SOLVABLE AS ODES

This happens if a PDE involves derivatives with respect to one variable only (or can be transformed to such a form), so that the other variable(s) can be treated as parameter(s). Solve for $u = u(x, y)$:

16. $u_{yy} = 0$ 17. $u_{xx} + 16\pi^2 u = 0$ 25. **System of PDEs.** Solve $u_{xx} = 0, u_{yy} = 0$

18. $25u_{yy} - 4u = 0$ 19. $u_y + y^2 u = 0$

20. $2u_{xx} + 9u_x + 4u = -3 \cos x - 29 \sin x$

21. $u_{yy} + 6u_y + 13u = 4e^{3y}$

22. $u_{xy} = u_x$ 23. $x^2 u_{xx} + 2xu_x - 2u = 0$

24. **Surface of revolution.** Show that the solutions $z = z(x, y)$ of $y z_x = x z_y$ represent surfaces of revolution. Give examples. *Hint.* Use polar coordinates r, θ and show that the equation becomes $z_\theta = 0$.

12.2 Modeling: Vibrating String, Wave Equation

In this section we model a vibrating string, which will lead to our first important PDE, that is, equation (3) which will then be solved in Sec. 12.3. *The student should pay very close attention to this delicate modeling process and detailed derivation starting from scratch,* as the skills learned can be applied to modeling other phenomena in general and in particular to modeling a vibrating membrane (Sec. 12.7).

We want to derive the PDE modeling small transverse vibrations of an elastic string, such as a violin string. We place the string along the x -axis, stretch it to length L , and fasten it at the ends $x = 0$ and $x = L$. We then distort the string, and at some instant, call it $t = 0$, we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at any time $t > 0$; see Fig. 286.

$u(x, t)$ will be the solution of a PDE that is the model of our physical system to be derived. This PDE should not be too complicated, so that we can solve it. Reasonable simplifying assumptions (just as for ODEs modeling vibrations in Chap. 2) are as follows.

Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Under these assumptions we may expect solutions $u(x, t)$ that describe the physical reality sufficiently well.

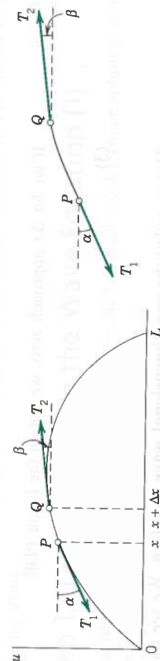


Fig. 286. Deflected string at fixed time t . Explanation on p. 544

where f^* is the odd periodic extension of f with the period $2L$ (Fig. 289). Since the initial deflection $f(x)$ is continuous on the interval $0 \leq x \leq L$ and zero at the endpoints, it follows from (17) that $u(x, t)$ is a continuous function of both variables x and t for all values of the variables. By differentiating (17) we see that $u(x, t)$ is a solution of (1), provided $f(x)$ is twice differentiable on the interval $0 < x < L$, and has one-sided second derivatives at $x = 0$ and $x = L$, which are zero. Under these conditions $u(x, t)$ is established as a solution of (1), satisfying (2) and (3) with $g(x) \equiv 0$.



Fig. 289. Odd periodic extension of $f(x)$

Generalized Solution. If $f'(x)$ and $f''(x)$ are merely piecewise continuous (see Sec. 6.1), or if those one-sided derivatives are not zero, then for each t there will be finitely many values of x at which the second derivatives of u appearing in (1) do not exist. Except at these points the wave equation will still be satisfied. We may then regard $u(x, t)$ as a “generalized solution,” as it is called, that is, as a solution in a broader sense. For instance, a triangular initial deflection as in Example 1 (below) leads to a generalized solution.

Physical Interpretation of the Solution (17). The graph of $f^*(x - ct)$ is obtained from the graph of $f^*(x)$ by shifting the latter ct units to the right (Fig. 290). This means that $f^*(x - ct)(c > 0)$ represents a wave that is traveling to the right as t increases. Similarly, $f^*(x + ct)$ represents a wave that is traveling to the left, and $u(x, t)$ is the superposition of these two waves.

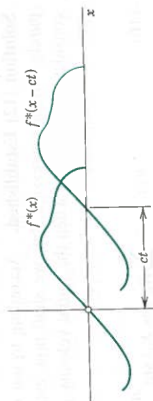


Fig. 290. Interpretation of (17)

EXAMPLE 1 Vibrating String if the Initial Deflection Is Triangular

Find the solution of the wave equation (1) satisfying (2) and corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$$

and initial velocity zero. (Figure 291 shows $f(x) = u(x, 0)$ at the top.)

Solution. Since $g(x) \equiv 0$, we have $B_n^* = 0$ in (12), and from Example 4 in Sec. 11.3 we see that the B_n are given by (5). Sec. 11.3. Thus (12) takes the form

$$u(x, t) = \frac{8k}{\pi^2} \left[\frac{1}{2} \sin \frac{\pi x}{L} \cos \frac{\pi c}{L} t - \frac{1}{3^2} \sin \frac{3\pi x}{L} \cos \frac{3\pi c}{L} t + \dots \right]$$

For graphing the solution we may use $u(x, 0) = f(x)$ and the above interpretation of the two functions in the representation (17). This leads to the graph shown in Fig. 291.

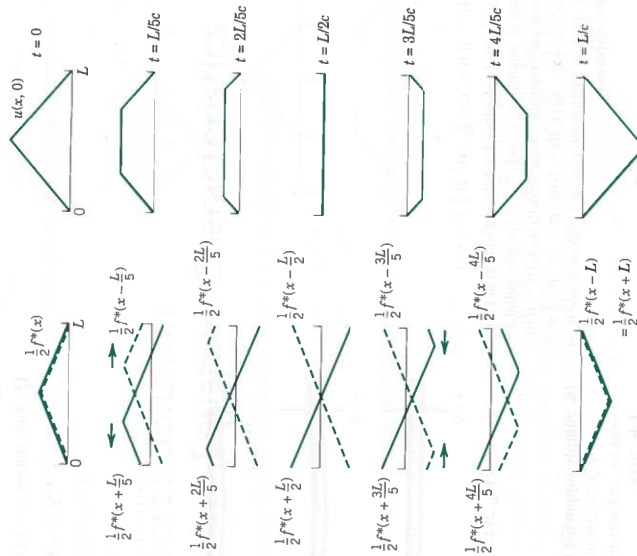


Fig. 291. Solution $u(x, t)$ in Example 1 for various values of t (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)

PROBLEM SET 12.3

1. **Frequency.** How does the frequency of the fundamental mode of the vibrating string depend on the length of the string? On the mass per unit length? What happens if we double the tension? Why is a contrabass larger than a violin?
2. **Physical Assumptions.** How would the motion of the string change if Assumption 3 were violated? Assumption 2? The second part of Assumption 1? The first part? Do we really need all these assumptions?
3. **String of length π .** Write down the derivation in this section for length $L = \pi$, to see the very substantial simplification of formulas in this case that may show ideas more clearly.

4. **CAS PROJECT. Graphing Normal Modes.** Write a program for graphing u_n with $L = \pi$ and c^2 of your choice similarly as in Fig. 287. Apply the program to u_2, u_3, u_4 . Also graph these solutions as surfaces over the xy -plane. Explain the connection between these two kinds of graphs.

5-13 DEFLECTION OF THE STRING

Find $u(x, t)$ for the string of length $L = 1$ and $c^2 = 1$ when the initial velocity is zero and the initial deflection with small k (say, 0.01) is as follows. Sketch or graph $u(x, t)$ as in Fig. 291 in the text.

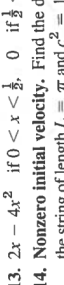
5. $k \sin 3\pi x$
6. $k(\sin \pi x - \frac{1}{2} \sin 2\pi x)$

- 7.
- $kx(1-x)$

- 9.
- 

- 10.
- 

- 11.
- 

- 12.
- 

- 13.
- $2x - 4x^2$
- if
- $0 < x < \frac{1}{2}$
- , 0 if
- $\frac{1}{2} < x < 1$

14. Nonzero initial velocity. Find the deflection
- $u(x, t)$
- of the string of length
- $L = \pi$
- and
- $c^2 = 1$
- for zero initial displacement and "triangular" initial velocity
- $u_t(x, 0) = 0.01x$
- if
- $0 \leq x \leq \frac{1}{2}\pi$
- ,
- $u_t(x, 0) = 0.01(\pi - x)$
- if
- $\frac{1}{2}\pi \leq x \leq \pi$
- . (Initial conditions with
- $u_t(x, 0) \neq 0$
- are hard to realize experimentally.)

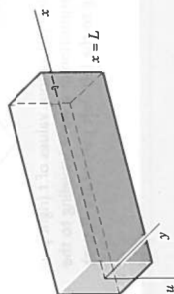


Fig. 292. Elastic beam

15-20 SEPARATION OF A FOURTH-ORDER PDE. VIBRATING BEAM

By the principles used in modeling the string it can be shown that small free vertical vibrations of a uniform elastic beam (Fig. 292) are modeled by the fourth-order PDE

$$(21) \quad \frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \quad (\text{Ref. [C11]})$$

where $c^2 = EI/\rho A$ (E = Young's modulus of elasticity, I = moment of inertia of the cross section with respect to the

y -axis in the figure, ρ = density, A = cross-sectional area). (Bending of a beam under a load is discussed in Sec. 3.3.)

15. Substituting
- $u = F(x)G(t)$
- into (21), show that

$$F^{(4)}/F = -\ddot{G}/c^2 G = \beta^4 = \text{const.}$$

$$F(x) = A \cos \beta x + B \sin \beta x$$

$$+ C \cosh \beta x + D \sinh \beta x,$$

$$G(t) = a \cos c\beta^2 t + b \sin c\beta^2 t.$$

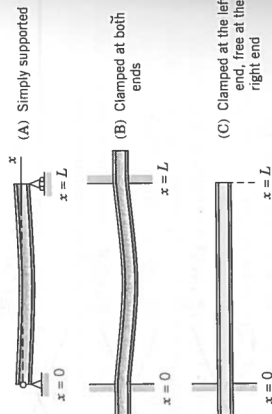


Fig. 293. Supports of a beam

16. Simply supported beam in Fig. 293A. Find solutions
- $u_n = F_n(x)G_n(t)$
- of (21) corresponding to zero initial velocity and satisfying the boundary conditions (see Fig. 293A)

$$u(0, t) = 0, u(L, t) = 0$$

(ends simply supported for all times t),

$$u_{xx}(0, t) = 0, u_{xx}(L, t) = 0$$

(zero moments, hence zero curvature, at the ends).

17. Find the solution of (21) that satisfies the conditions in Prob. 16 as well as the initial condition

$$u(x, 0) = f(x) = x(L - x).$$

18. Compare the results of Probs. 17 and 7. What is the basic difference between the frequencies of the normal modes of the vibrating string and the vibrating beam?

19. Clamped beam in Fig. 293B. What are the boundary conditions for the clamped beam in Fig. 293B? Show that
- F
- in Prob. 15 satisfies these conditions if
- βL
- is a solution of the equation

$$(22) \quad \cosh \beta L \cos \beta L = 1.$$

Determine approximate solutions of (22), for instance, graphically from the intersections of the curves of $\cos \beta L$ and $1/\cosh \beta L$.

20. Clamped-free beam in Fig. 293C. If the beam is clamped at the left and free at the right (Fig. 293C), the boundary conditions are

$$(23) \quad u_x(0, t) = 0, \quad \cosh \beta L \cos \beta L = -1.$$

Find approximate solutions of (23).

$$u_{xx}(L, t) = 0, \quad u_{xxx}(L, t) = 0.$$

12.4 D'Alembert's Solution of the Wave Equation. Characteristics

It is interesting that the solution (17), Sec. 12.3, of the wave equation

$$(1) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho},$$

can be immediately obtained by transforming (1) in a suitable way, namely, by introducing the new independent variables

$$(2) \quad v = x + ct, \quad w = x - ct.$$

Then u becomes a function of v and w . The derivatives in (1) can now be expressed in terms of derivatives with respect to v and w by the use of the chain rule in Sec. 9.6. Denoting partial derivatives by subscripts, we see from (2) that $v_x = 1$ and $w_x = 1$. For simplicity let us denote $u(x, t)$, as a function of v and w , by the same letter u . Then

$$u_x = u_v v_x + u_w w_x = u_v + u_w.$$

We now apply the chain rule to the right side of this equation. We assume that all the partial derivatives involved are continuous, so that $u_{vw} = u_{wv}$. Since $v_x = 1$ and $w_x = 1$, we obtain

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}.$$

Transforming the other derivative in (1) by the same procedure, we find

$$u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww}).$$

By inserting these two results in (1) we get (see footnote 2 in App. A3.2)

$$(3) \quad u_{vw} = \frac{\partial^2 u}{\partial v \partial w} = 0.$$

The point of the present method is that (3) can be readily solved by two successive integrations, first with respect to w and then with respect to v . This gives

$$\frac{\partial u}{\partial v} = h(v) \quad \text{and} \quad u = \int h(v) dv + \psi(w).$$

12.5 Modeling: Heat Flow from a Body in Space. Heat Equation

Type	New Variables	Normal Form
Hyperbolic	$v = \Phi$	$u_{vv} = F_1$
Parabolic	$v = x$	$u_{vw} = F_2$
Elliptic	$v = \frac{1}{2}(\Phi + \Psi)$ $w = \frac{1}{2}(\Phi - \Psi)$	$u_{vv} + u_{ww} = F_3$

Here, $\Phi = \Phi(x, y, z)$, $\Psi = \Psi(x, y, z)$, $F_1 = F_1(x, y, z)$, $F_2 = F_2(x, y, z)$, and we denote u as function of v, w again by u for simplicity. We see that the normal form of a hyperbolic PDE is as in d'Alembert's solution. In the parabolic case we get just one family of solutions $\Phi = \Psi$. In the elliptic case, $i = \sqrt{-1}$, and the characteristics are complex and are of minor interest. For derivation, see Ref. [GenRef3] in App. 1.

EXAMPLE 1 D'Alembert's Solution Obtained Systematically

The theory of characteristics gives d'Alembert's solution in a systematic fashion. To see this, we write the wave equation $u_{tt} - c^2 u_{xx} = 0$ in the form (14) by setting $y = ct$. By the chain rule, $u_t = u_y y_t = cu_y$ and $u_{tt} = c^2 u_{yy}$. Division by c^2 gives $u_{xx} - u_{yy} = 0$, as stated before. Hence the characteristic equation is $y^2 - 1 = (y' + 1)(y' - 1) = 0$. The two families of solutions (characteristics) are $\Phi(x, y) = y + x = \text{const}$ and $\Psi(x, y) = y - x = \text{const}$. This gives the new variables $v = \Phi = y + x = ct + x$ and $w = \Psi = y - x = ct - x$ and d'Alembert's solution $u = f_1(x + ct) + f_2(x - ct)$.

PROBLEM SET 12.4

- Show that c is the speed of each of the two waves given by (4).
- Show that, because of the boundary conditions (2), Sec. 12.3, the function f in (13) of this section must be odd and of period $2L$.
- If a steel wire 2 m in length weighs 0.9 nt (about 0.20 lb) and is stretched by a tensile force of 300 nt (about 67.4 lb), what is the corresponding speed of transverse waves?
- What are the frequencies of the eigenfunctions in Prob. 3?
3. GRAPHING SOLUTIONS
- ing (13) sketch or graph a figure (similar to Fig. 291 in 12.3) of the deflection $u(x, t)$ of a vibrating string of length $L = 1$, ends fixed, $c = 1$ starting with initial velocity 0 and initial deflection (k small, say, $k = 0.01$).
- $f(x) = k \sin \pi x$ 6. $f(x) = k(1 - \cos \pi x)$
- $f(x) = k \sin 2\pi x$ 8. $f(x) = kx(1 - x)$
18. NORMAL FORMS
- nd the type, transform to normal form, and solve. Show ur work in detail.
1. $u_{xx} + 4u_{yy} = 0$ 10. $u_{xx} - 16u_{yy} = 0$
11. $u_{xx} + 2u_{xy} + u_{yy} = 0$ 12. $u_{xx} - 2u_{xy} + u_{yy} = 0$
13. $u_{xx} + 5u_{xy} + 4u_{yy} = 0$ 14. $xu_{xy} - yu_{yy} = 0$
15. $xu_{xx} - yu_{xy} = 0$ 16. $u_{xx} + 2u_{xy} + 10u_{yy} = 0$
17. $u_{xx} - 4u_{xy} + 5u_{yy} = 0$ 18. $u_{xx} - 6u_{xy} + 9u_{yy} = 0$
19. Longitudinal Vibrations of an Elastic Bar or Rod.
- These vibrations in the direction of the x -axis are modeled by the wave equation $u_{tt} = c^2 u_{xx}$, $c^2 = E/\rho$ (see Tolstov [C9], p. 275). If the rod is fastened at one end, $x = 0$, and free at the other, $x = L$, we have $u(0, t) = 0$ and $u_x(L, t) = 0$. Show that the motion corresponding to initial displacement $u(x, 0) = f(x)$ and initial velocity zero is
- $$u = \sum_{n=0}^{\infty} A_n \sin p_n x \cos p_n ct,$$
- $$A_n = \frac{2}{L} \int_0^L f(x) \sin p_n x \, dx, \quad p_n = \frac{(2n+1)\pi}{2L}.$$

20. Tricomi and Airy equations.² Show that the Tricomi equation $y u_{xx} + u_{yy} = 0$ is of mixed type. Obtain the Airy equation $G'' - yG = 0$ from the Tricomi equation by separation. (For solutions, see p. 446 of Ref. [GenRef1] listed in App. 1.)

²Sir GEORGE BIDELE AIRY (1801–1892), English mathematician, known for his work in elasticity, FRANCESCO TRICOMI (1897–1978), Italian mathematician, who worked in integral equations and functional analysis.

After the wave equation (Sec. 12.2) we now derive and discuss the next “big” PDE, the heat equation, which governs the temperature u in a body in space. We obtain this model of temperature distribution under the following.

Physical Assumptions

1. The specific heat σ and the density ρ of the material of the body are constant. No heat is produced or disappears in the body.
2. Experiments show that, in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient (cf. Sec. 9.7) of the temperature; that is, the velocity $\mathbf{v}$ of the heat flow in the body is of the form

$$\mathbf{v} = -K \text{ grad } u \quad (1)$$

- where $u(x, y, z, t)$ is the temperature at a point (x, y, z) and time t .
3. The thermal conductivity K is constant, as is the case for homogeneous material and nonextreme temperatures.

Under these assumptions we can model heat flow as follows. Let T be a region in the body bounded by a surface S with outer unit normal vector $\mathbf{n}$ such that the divergence theorem (Sec. 10.7) applies. Then

$$\mathbf{v} \cdot \mathbf{n}$$

is the component of $\mathbf{v}$ in the direction of $\mathbf{n}$. Hence $|\mathbf{v} \cdot \mathbf{n} \Delta A|$ is the amount of heat leaving T (if $\mathbf{v} \cdot \mathbf{n} > 0$ at some point P) or entering T (if $\mathbf{v} \cdot \mathbf{n} < 0$ at P) per unit time at some point P of S through a small portion ΔS of S of area ΔA . Hence the total amount of heat that flows across S from T is given by the surface integral

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, dA.$$

Note that, so far, this parallels the derivation on fluid flow in Example 1 of Sec. 10.8. Using Gauss's theorem (Sec. 10.7), we now convert our surface integral into a volume integral over the region T . Because of (1) this gives [use (3) in Sec. 9.8]

$$\begin{aligned} (2) \quad \iint_S \mathbf{v} \cdot \mathbf{n} \, dA &= -K \iint_S (\text{grad } u) \cdot \mathbf{n} \, dA = -K \iiint_T \text{div} (\text{grad } u) \, dx \, dy \, dz \\ &= -K \iiint_T \nabla^2 u \, dx \, dy \, dz. \end{aligned}$$

Here,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

is the Laplacian of u .