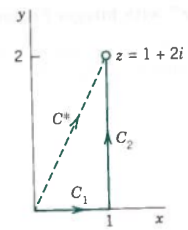


Fig. 343. Paths in Example 7

(b) We now have

$$C_1: z(t) = t, \quad \dot{z}(t) = 1, \quad f(z(t)) = x(t) = t \quad (0 \leqq t \leqq 1)$$

$$C_2: z(t) = 1 + it, \quad \dot{z}(t) = i, \quad f(z(t)) = x(t) = 1 \quad (0 \leqq t \leqq 2).$$

Using (6) we calculate

$$\int_C \operatorname{Re} z\, dz = \int_{C_1} \operatorname{Re} z\, dz + \int_{C_2} \operatorname{Re} z\, dz = \int_0^1 t\, dt + \int_0^2 1 \cdot i\, dt = \frac{1}{2} + 2i.$$

Note that this result differs from the result in (a). ■

Bounds for Integrals. *ML*-Inequality

There will be a frequent need for estimating the absolute value of complex line integrals. The basic formula is

$$\left| \int_C f(z)\, dz \right| \leqq ML \qquad \textit{(ML-inequality)}; \tag{13}$$

L is the length of C and M a constant such that $|f(z)| \leqq M$ everywhere on C.

PROOF Taking the absolute value in (2) and applying the generalized inequality (6*) in Sec. 13.2, we obtain

$$|S_n| = \left| \sum_{m=1}^{n} f(\zeta_m)\, \Delta z_m \right| \leqq \sum_{m=1}^{n} |f(\zeta_m)||\Delta z_m| \leqq M \sum_{m=1}^{n} |\Delta z_m|.$$

Now $|\Delta z_m|$ is the length of the chord whose endpoints are z_{m-1} and z_m (see Fig. 340). Hence the sum on the right represents the length L^* of the broken line of chords whose endpoints are $z_0, z_1, \cdots, z_n (= Z)$. If n approaches infinity in such a way that the greatest $|\Delta t_m|$ and thus $|\Delta z_m|$ approach zero, then L^* approaches the length L of the curve C, by the definition of the length of a curve. From this the inequality (13) follows. ■

We cannot see from (13) how close to the bound ML the actual absolute value of the integral is, but this will be no handicap in applying (13). For the time being we explain the practical use of (13) by a simple example.

EXAMPLE 8 **Estimation of an Integral**

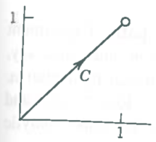


Fig. 344. Path in Example 8

Find an upper bound for the absolute value of the integral

$$\int_C z^2\, dz, \qquad C \text{ the straight-line segment from 0 to } 1 + i, \text{ Fig. 344.}$$

Solution. $L = \sqrt{2}$ and $|f(z)| = |z^2| \leqq 2$ on C gives by (13)

$$\left| \int_C z^2\, dz \right| \leqq 2\sqrt{2} = 2.8284.$$

The absolute value of the integral is $|-\frac{2}{3} + \frac{2}{3}i| = \frac{2}{3}\sqrt{2} = 0.9428$ (see Example 1). ■

Summary on Integration. Line integrals of $f(z)$ can always be evaluated by (10), using a representation (1) of the path of integration. If $f(z)$ is analytic, indefinite integration by (9) as in calculus will be simpler (proof in the next section).

PROBLEM SET 14.1

1–10 FIND THE PATH and sketch it.

1. $z(t) = (1 + \frac{1}{4}i)t, \quad (1 \leqq t \leqq 6)$
2. $z(t) = 3 + i + (1 - i)t, \quad (0 \leqq t \leqq 3)$
3. $z(t) = t + 4t^2 i, \quad (0 \leqq t \leqq 1)$
4. $z(t) = t + (1 - t)^2 i, \quad (-1 \leqq t \leqq 1)$
5. $z(t) = 2 - 2i + \sqrt{5}e^{-it}, \quad (0 \leqq t \leqq 2\pi)$
6. $z(t) = 1 + i + e^{-\pi i t}, \quad (0 \leqq t \leqq 2)$
7. $z(t) = 1 + 2e^{\pi i t/4}, \quad (0 \leqq t \leqq 2)$
8. $z(t) = 5e^{-it}, \quad (0 \leqq t \leqq \pi/2)$
9. $z(t) = t + i(1 - t)^3, \quad (-2 \leqq t \leqq 2)$
10. $z(t) = 2\cos t + i \sin t, \quad (0 \leqq t \leqq 2\pi)$

11–20 FIND A PARAMETRIC REPRESENTATION and sketch the path.

11. Segment from $(-1, 2)$ to $(1, 4)$
12. From $(0, 0)$ to $(2, 1)$ along the axes
13. Upper half of $|z - 4 + i| = 4$ from $(5, -1)$ to $(-3, -1)$
14. Unit circle, clockwise
15. $4x^2 - y^2 = 4$, the branch through $(0, 2)$
16. Ellipse $4x^2 + 9y^2 = 36$, counterclockwise
17. $|z + a - ib| = r$, clockwise
18. $y = 1/x$ from $(1, 1)$ to $(5, \frac{1}{5})$
19. Parabola $y = 1 - \frac{1}{4}x^2, \quad (-2 \leqq x \leqq 2)$
20. $4(x - 2)^2 + 5(y + 1)^2 = 20$

21–30 INTEGRATION

Integrate by the first method or state why it does not apply and use the second method. Show the details.

21. $\int_C \operatorname{Re} z\, dz$, C the shortest path from $1 + i$ to $5 + 5i$
22. $\int_C \operatorname{Re} z\, dz$, C the parabola $y = 1 + \frac{1}{2}(x - 1)^2$ from $1 + i$ to $3 + 3i$
23. $\int_C e^z\, dz$, C the shortest path from $\pi/2i$ to πi
24. $\int_C \cos 2z\, dz$, C the semicircle $|z| = \pi, x \geqq 0$ from $-\pi i$ to πi
25. $\int_C z \exp(z^2)\, dz$, C from 1 along the axes to i
26. $\int_C (z + z^{-1})\, dz$, C the unit circle, counterclockwise
27. $\int_C \sec^2 z\, dz$, any path from $\pi/4$ to $\pi i/4$
28. $\int_C \left(\frac{5}{z - 2i} - \frac{6}{(z - 2i)^2}\right) dz$, C the circle $|z - 2i| = 4$, clockwise
29. $\int_C \operatorname{Im} z^2\, dz$ counterclockwise around the triangle with vertices $0, 1, i$
30. $\int_C \operatorname{Re} z^2\, dz$ clockwise around the boundary of the square with vertices $0, i, 1 + i, 1$
31. **CAS PROJECT. Integration.** Write programs for the two integration methods. Apply them to problems of your choice. Could you make them into a joint program that also decides which of the two methods to use in a given case?

Also, if $G'(z) = f(z)$, then $F'(z) - G'(z) \equiv 0$ in D; hence $F(z) - G(z)$ is constant in D (see Team Project 30 in Problem Set 13.4). That is, two indefinite integrals of $f(z)$ can differ only by a constant. The latter drops out in (9) of Sec. 14.1, so that we can use any indefinite integral of $f(z)$. This proves Theorem 3. ∎

Cauchy's Integral Theorem for Multiply Connected Domains

Cauchy's theorem applies to multiply connected domains. We first explain this for a **doubly connected domain** D with outer boundary curve C_1 and inner C_2 (Fig. 353). If a function $f(z)$ is analytic in any domain D^* that contains D and its boundary curves, we claim that

$$\oint_{C_1} f(z)\, dz = \oint_{C_2} f(z)\, dz \qquad \text{(Fig. 353)} \tag{6}$$

both integrals being taken counterclockwise (or both clockwise, and regardless of whether or not the full interior of C_2 belongs to D^*).

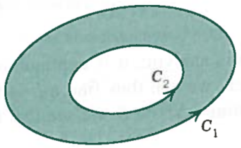


Fig. 353. Paths in (5)

PROOF By two cuts $\tilde{C}_1$ and $\tilde{C}_2$ (Fig. 354) we cut D into two simply connected domains D_1 and D_2 in which and on whose boundaries $f(z)$ is analytic. By Cauchy's integral theorem the integral over the entire boundary of D_1 (taken in the sense of the arrows in Fig. 354) is zero, and so is the integral over the boundary of D_2, and thus their sum. In this sum the integrals over the cuts $\tilde{C}_1$ and $\tilde{C}_2$ cancel because we integrate over them in both directions—this is the key—and we are left with the integrals over C_1 (counterclockwise) and C_2 (clockwise; see Fig. 354); hence by reversing the integration over C_2 (to counterclockwise) we have

$$\oint_{C_1} f\, dz - \oint_{C_2} f\, dz = 0$$

and (6) follows. ∎

For domains of higher connectivity the idea remains the same. Thus, for a **triply connected domain** we use three cuts $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ (Fig. 355). Adding integrals as before, the integrals over the cuts cancel and the sum of the integrals over C_1 (counterclockwise) and C_2, C_3 (clockwise) is zero. Hence the integral over C_1 equals the sum of the integrals over C_2 and C_3, all three now taken counterclockwise. Similarly for quadruply connected domains, and so on.

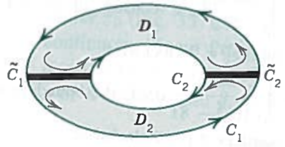


Fig. 354. Doubly connected domain

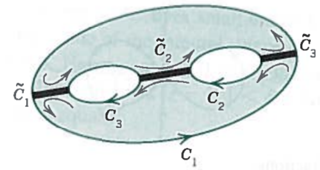


Fig. 355. Triply connected domain

PROBLEM SET 14.2

1–8 COMMENTS ON TEXT AND EXAMPLES

1. **Cauchy's Integral Theorem.** Verify Theorem 1 for the integral of z^2 over the boundary of the square with vertices $\pm 1 \pm i$. *Hint.* Use deformation.
2. For what contours C will it follow from Theorem 1 that
(a) $\displaystyle\int_C \frac{dz}{z-1} = 0$, (b) $\displaystyle\int_C \frac{\exp(1/z^2)}{z^2+4} = 0$
3. **Deformation principle.** Can we conclude from Example 4 that the integral is also zero over the contour in Prob. 1?
4. If the integral of a function over the unit circle equals 2 and over the circle of radius 3 equals 6, can the function be analytic everywhere in the annulus $1 < |z| < 3$?
5. **Connectedness.** What is the connectedness of the domain in which $(\cos z^2)/(z^4 + 1)$ is analytic?
6. **Path independence.** Verify Theorem 2 for the integral of e^z from 0 to $1 + i$ (a) over the shortest path and (b) over the x-axis to 1 and then straight up to $1 + i$.
7. **Deformation.** Can we conclude in Example 2 that the integral of $1/(z^2 + 4)$ over (a) $|z - 2| = 2$ and (b) $|z - 2| = 3$ is zero?
8. **TEAM EXPERIMENT. Cauchy's Integral Theorem.**
(a) Main Aspects. Each of the problems in Examples 1–5 explains a basic fact in connection with Cauchy's theorem. Find five examples of your own, more complicated ones if possible, each illustrating one of those facts.
(b) Partial fractions. Write $f(z)$ in terms of partial fractions and integrate it counterclockwise over the unit circle, where
(i) $f(z) = \dfrac{2z + 3i}{z^2 + \frac{1}{4}}$ (ii) $f(z) = \dfrac{z+1}{z^2 + 2z}$.
(c) Deformation of path. Review (c) and (d) of Team Project 34, Sec. 14.1, in the light of the principle of deformation of path. Then consider another family of paths with common endpoints, say, $z(t) = t + ia(t - t^2)$, $0 \le t \le 1$, a a real constant, and experiment with the integration of analytic and nonanalytic functions of your choice over these paths (e.g., z, $\text{Im } z$, z^2, $\text{Re } z^2$, $\text{Im } z^2$, etc.).

9–19 CAUCHY'S THEOREM APPLICABLE?

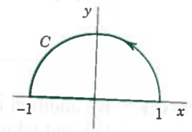
Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

9. $f(z) = \exp(z^2)$
10. $f(z) = \tan \frac{1}{4} z$
11. $f(z) = 1/(4z - 1)$
12. $f(z) = \bar{z}^3$
13. $f(z) = 1/(z^4 - 1.2)$
14. $f(z) = 1/\bar{z}$
15. $f(z) = \text{Re } z$
16. $f(z) = 1/(\pi z - 1)$
17. $f(z) = 1/|z|^2$
18. $f(z) = 1/(5z - 1)$
19. $f(z) = z^3 \cot z$

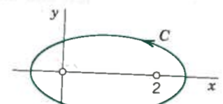
20–30 FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

20. $\oint_C \text{Ln}(1 - z)\, dz$, C the boundary of the parallelogram with vertices $\pm i, \pm(1 + i)$.
21. $\oint_C \dfrac{dz}{z - 2i}$, C the circle $|z| = \pi$ counterclockwise.
22. $\oint_C \text{Re } z\, dz$, C:

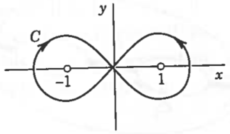


23. $\oint_C \dfrac{2z - 1}{z^2 - z}\, dz$, C:



Use partial fractions.

24. $\oint_C \frac{dz}{z^2-1}$, C:



Use partial fractions.

25. $\oint_C \frac{e^z}{z}\,dz$, C consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

26. $\oint_C \coth \tfrac{1}{2}z\,dz$, C the circle $|z - \tfrac{1}{2}\pi i| = 1$ clockwise.

27. $\oint_C \frac{\cos z}{z}\,dz$, C consists of $|z| = 1$ counterclockwise and $|z| = 3$ clockwise.

28. $\oint_C \frac{\tan \frac{1}{2}z}{16z^4 - 81}\,dz$, C the boundary of the square with vertices $\pm 1, \pm i$ clockwise.

29. $\oint_C \frac{\sin z}{z + 4iz}\,dz$, C: $|z - 4 - 2i| = 6.5$.

30. $\oint_C \frac{2z^3 + z^2 + 4}{z^4 + 4z^2}\,dz$, C: $|z - 2| = 4$ clockwise. Use partial fractions.

14.3 Cauchy's Integral Formula

Cauchy's integral theorem leads to Cauchy's integral formula. This formula is useful for evaluating integrals as shown in this section. It has other important roles, such as in proving the surprising fact that analytic functions have derivatives of all orders, as shown in the next section, and in showing that all analytic functions have a Taylor series representation (to be seen in Sec. 15.4).

THEOREM 1 **Cauchy's Integral Formula**

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig. 356),

(1) $$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0) \qquad \textbf{(Cauchy's integral formula)}$$

the integration being taken counterclockwise. Alternatively (for representing $f(z_0)$ by a contour integral, divide (1) by $2\pi i$),

(1*) $$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz \qquad \textbf{(Cauchy's integral formula).}$$

PROOF By addition and subtraction, $f(z) = f(z_0) + [f(z) - f(z_0)]$. Inserting this into (1) on the left and taking the constant factor $f(z_0)$ out from under the integral sign, we have

(2) $$\oint_C \frac{f(z)}{z - z_0}\,dz = f(z_0)\oint_C \frac{dz}{z - z_0} + \oint_C \frac{f(z) - f(z_0)}{z - z_0}\,dz.$$

The first term on the right equals $f(z_0) \cdot 2\pi i$, which follows from Example 6 in Sec. 14.2 with $m = -1$. If we can show that the second integral on the right is zero, then it would prove the theorem. Indeed, we can. The integrand of the second integral is analytic, except at z_0. Hence, by (6) in Sec. 14.2, we can replace C by a small circle K of radius ρ and center z_0 (Fig. 357), without altering the value of the integral. Since $f(z)$ is analytic, it is continuous (Team Project 24, Sec. 13.3). Hence, an $\epsilon > 0$ being given, we can find a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all z in the disk $|z - z_0| < \delta$. Choosing the radius ρ of K smaller than δ, we thus have the inequality

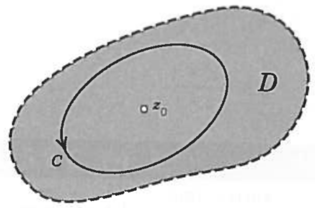


Fig. 356. Cauchy's integral formula

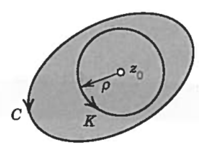


Fig. 357. Proof of Cauchy's integral formula

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| < \frac{\epsilon}{\rho}$$

at each point of K. The length of K is $2\pi\rho$. Hence, by the ML-inequality in Sec. 14.1,

$$\left|\oint_K \frac{f(z) - f(z_0)}{z - z_0}\,dz\right| < \frac{\epsilon}{\rho}2\pi\rho = 2\pi\epsilon.$$

Since ϵ (> 0) can be chosen arbitrarily small, it follows that the last integral in (2) must have the value zero, and the theorem is proved. ■

EXAMPLE 1 **Cauchy's Integral Formula**

$$\oint_C \frac{e^z}{z - 2}\,dz = 2\pi i e^z\Big|_{z=2} = 2\pi i e^2 = 46.4268i$$

for any contour enclosing $z_0 = 2$ (since e^z is entire), and zero for any contour for which $z_0 = 2$ lies outside (by Cauchy's integral theorem). ■

EXAMPLE 2 **Cauchy's Integral Formula**

$$\oint_C \frac{z^3 - 6}{2z - i}\,dz = \oint_C \frac{\frac{1}{2}z^3 - 3}{z - \frac{1}{2}i}\,dz$$

$$= 2\pi i[\tfrac{1}{2}z^3 - 3]|_{z=i/2}$$

$$= \frac{\pi}{8} - 6\pi i \qquad (z_0 = \tfrac{1}{2}i \text{ inside } C).$$ ■

EXAMPLE 3 **Integration Around Different Contours**

Integrate

$$g(z) = \frac{z^2 + 1}{z^2 - 1} = \frac{z^2 + 1}{(z + 1)(z - 1)}$$

counterclockwise around each of the four circles in Fig. 358.