PROBLEM SET 15.3

1. **Relation to Calculus.** Material in this section generalizes calculus. Give details.

2. **Termwise addition.** Write out the details of the proof on termwise addition and subtraction of power series.

3. **On Theorem 3.** Prove that $\sqrt[n]{n} \to 1$ as $n \to \infty$, as claimed.

4. **Cauchy product.** Show that $(1 - z)^{-2} = \sum_{n=0}^{\infty} (n + 1)z^n$ (a) by using the Cauchy product, (b) by differentiating a suitable series.

5–15 RADIUS OF CONVERGENCE BY DIFFERENTIATION OR INTEGRATION

Find the radius of convergence in two ways: (a) directly by the Cauchy–Hadamard formula in Sec. 15.2, and (b) from a series of simpler terms by using Theorem 3 or Theorem 4.

5. $\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z - 2i)^n$

6. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{z}{2\pi}\right)^{2n+1}$

7. $\sum_{n=1}^{\infty} \frac{n}{5^n}(z + 2i)^{2n}$

8. $\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n$

9. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n(n+1)(n+2)} z^{2n}$

10. $\sum_{n=k}^{\infty} \binom{n}{k}\left(\frac{z}{2}\right)^n$

11. $\sum_{n=1}^{\infty} \frac{2^n n(n+1)}{5^n} z^{2n}$

12. $\sum_{n=1}^{\infty} \frac{2n(2n-1)}{n^n} z^{2n-2}$

13. $\sum_{n=0}^{\infty} \left[\binom{n+k}{k}\right]^{-1} z^{n+k}$

14. $\sum_{n=0}^{\infty} \binom{n+m}{m} z^n$

15. $\sum_{n=2}^{\infty} \frac{5^n n(n-1)}{3^n}(z - i)^n$

16–20 APPLICATIONS OF THE IDENTITY THEOREM

State clearly and explicitly where and how you are using Theorem 2.

16. **Even functions.** If $f(z)$ in (2) is *even* (i.e., $f(-z) = f(z)$), show that $a_n = 0$ for odd n. Give examples.

17. **Odd function.** If $f(z)$ in (2) is *odd* (i.e., $f(-z) = -f(z)$), show that $a_n = 0$ for even n. Give examples.

18. **Binomial coefficients.** Using $(1 + z)^p(1 + z)^q = (1 + z)^{p+q}$, obtain the basic relation

$$\sum_{n=0}^{r} \binom{p}{n}\binom{q}{r-n} = \binom{p+q}{r}.$$

19. Find applications of Theorem 2 in differential equations and elsewhere.

20. **TEAM PROJECT. Fibonacci numbers.**[2] (a) The Fibonacci numbers are recursively defined by $a_0 = a_1 = 1$, $a_{n+1} = a_n + a_{n-1}$ if $n = 1, 2, \cdots$. Find the limit of the sequence (a_{n+1}/a_n).

(b) Fibonacci's rabbit problem. Compute a list of $a_1, \cdots, a_{12}$. Show that $a_{12} = 233$ is the number of pairs of rabbits after 12 months if initially there is 1 pair and each pair generates 1 pair per month, beginning in the second month of existence (no deaths occurring).

(c) Generating function. Show that the *generating function* of the **Fibonacci numbers** is $f(z) = 1/(1 - z - z^2)$; that is, if a power series (1) represents this $f(z)$, its coefficients must be the Fibonacci numbers and conversely. *Hint.* Start from $f(z)(1 - z - z^2) = 1$ and use Theorem 2.

PROBLEM SET 15.4

1. Calculus. Which of the series in this section have you discussed in calculus? What is new?

2. On Examples 5 and 6. Give all the details in the derivation of the series in those examples.

3–10 MACLAURIN SERIES

Find the Maclaurin series and its radius of convergence.

3. $\sin \dfrac{z^2}{2}$

4. $\dfrac{z+2}{1-z^2}$

5. $\dfrac{1}{8+z^4}$

6. $\dfrac{1}{1+2iz}$

7. $2\sin^2(z/2)$

8. $\sin^2 z$

9. $\displaystyle\int_0^z \exp(-t^2)\,dt$

10. $\displaystyle\exp(z^2)\int_0^z \exp(-t^2)\,dt$

11–14 HIGHER TRANSCENDENTAL FUNCTIONS

Find the Maclaurin series by termwise integrating the integrand. (The integrals cannot be evaluated by the usual methods of calculus. They define the **error function** erf z, **sine integral** Si(z), and **Fresnel integrals**[4] S(z) and C(z), which occur in statistics, heat conduction, optics, and other applications. These are special so-called higher transcendental functions.)

11. $S(z) = \displaystyle\int_0^z \sin t^2\,dt$

12. $C(z) = \displaystyle\int_0^z \cos t^2\,dt$

13. $\operatorname{erf} z = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-t^2}\,dt$

14. $\operatorname{Si}(z) = \displaystyle\int_0^z \frac{\sin t}{t}\,dt$

15. CAS Project. sec, tan. (a) Euler numbers. The Maclaurin series

$$\text{(21)} \qquad \sec z = E_0 - \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 - + \cdots$$

defines the *Euler numbers* E_{2n}. Show that $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$. Write a program that computes the E_{2n} from the coefficient formula in (1) or extracts them as a list from the series. (For tables see Ref. [GenRef1], p. 810, listed in App. 1.)

(b) Bernoulli numbers. The Maclaurin series

$$\text{(22)} \qquad \frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots$$

defines the *Bernoulli numbers* B_n. Using undetermined coefficients, show that

$$\text{(23)} \qquad \begin{aligned} &B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_3 = 0,\\ &B_4 = -\tfrac{1}{30}, \quad B_5 = 0, \quad B_6 = \tfrac{1}{42}, \cdots. \end{aligned}$$

Write a program for computing B_n.

(c) Tangent. Using (1), (2), Sec. 13.6, and (22), show that $\tan z$ has the following Maclaurin series and calculate from it a table of $B_0, \cdots, B_{20}$:

$$\text{(24)} \qquad \tan z = \frac{2i}{e^{2iz}-1} - \frac{4i}{e^{4iz}-1} - i = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2^{2n}(2^{2n}-1)}{(2n)!}B_{2n}\,z^{2n-1}.$$

16. Inverse sine. Developing $1/\sqrt{1-z^2}$ and integrating, show that

$$\arcsin z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \cdots \ (|z| < 1).$$

Show that this series represents the principal value of $\arcsin z$ (defined in Team Project 30, Sec. 13.7).

17. TEAM PROJECT. Properties from Maclaurin Series. Clearly, from series we can compute function values. In this project we show that properties of functions can often be discovered from their Taylor or Maclaurin series. Using suitable series, prove the following.

(a) The formulas for the derivatives of e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$, and $\operatorname{Ln}(1+z)$

(b) $\frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$

(c) $\sin z \neq 0$ for all pure imaginary $z = iy \neq 0$

18–25 TAYLOR SERIES

Find the Taylor series with center z_0 and its radius of convergence.

18. $1/z$, $z_0 = i$

19. $1/(1+z)$, $z_0 = -i$

20. $\cos^2 z$, $z_0 = \pi/2$

21. $\cos z$, $z_0 = \pi$

22. $\cosh(z - \pi i)$, $z_0 = \pi i$

23. $1/(z-i)^2$, $z_0 = -i$

24. $e^{z(z-2)}$, $z_0 = 1$

25. $\sinh(2z - i)$, $z_0 = i/2$

[4] AUGUSTIN FRESNEL (1788–1827), French physicist and engineer, known for his work in optics.

PROBLEM SET 16.1

1–4 LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. Show the details of your work.

1. $\dfrac{\cos z}{z^4}$

2. $\dfrac{\exp(-1/z^2)}{z^2}$

3. $z^3 \cosh \dfrac{1}{z}$

4. $\dfrac{e^z}{z^2 - z^3}$

5–9 LAURENT SERIES NEAR A SINGULARITY AT z_0

Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. Show details.

5. $\dfrac{e^z}{(z-1)^2}$, $z_0 = 1$

6. $\dfrac{1}{z^2(z-i)}$, $z_0 = i$

7. $\dfrac{\sin z}{(z - \frac{1}{4}\pi)^3}$, $z_0 = \frac{1}{4}\pi$

8. **CAS PROJECT. Partial Fractions.** Write a program for obtaining Laurent series by the use of partial fractions. Using the program, verify the calculations in Example 5 of the text. Apply the program to two other functions of your choice.

9. **TEAM PROJECT. Laurent Series. (a) Uniqueness.** Prove that the Laurent expansion of a given analytic function in a given annulus is unique.

   (b) Accumulation of singularities. Does $\tan(1/z)$ have a Laurent series that converges in a region $0 < |z| < R$? (Give a reason.)

   (c) Integrals. Expand the following functions in a Laurent series that converges for $|z| > 0$:

$$\frac{1}{z^2}\int_0^z \frac{e^t - 1}{t}\,dt, \qquad \frac{1}{z^3}\int_0^z \frac{\sin t}{t}\,dt.$$

10–13 TAYLOR AND LAURENT SERIES

Find all Taylor and Laurent series with center z_0. Determine the precise regions of convergence. Show details.

10. $\dfrac{1}{1 - z^2}$, $z_0 = 0$

11. $\dfrac{1}{z}$, $z_0 = 1$

12. $\dfrac{1}{z^2}$, $z_0 = i$

13. $\dfrac{z^8}{1 - z^4}$, $z_0 = 0$