NTNU

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Department of Mathematical Sciences

Examination paper for **TMA4120 Calculus 4K**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: (Code C): Approved simple calculator.

Other information:

- Every answer must be justified; describe clearly how you have reached your answers.

Language: English

Number of pages: 8

Number of pages enclosed: 0

Checked by:

Date Signature

Problem 1 Find a solution $y(t)$ for the following equation for $t > 0$ by using the Laplace transform:

$$y'(t) + y(t) = f(t), \quad f(t) = \begin{cases} 0, & 0 < t < 2, \\ t, & 2 < t, \end{cases} \qquad y(0) = 2.$$

Problem 2 Let $f(x)$ be a π-periodic function on $\mathbb{R}$, defined by

$$f(x) = \max\left\{0, x - \frac{\pi}{2}\right\}, \quad \text{for } 0 < x < \pi.$$

(i) Sketch this Fourier series on the interval $-\pi < x < \pi$. (ii) Find the Fourier series of $f(x)$. (iii) What is the value of this Fourier series at $x = \pi$?

Hint: you may use $\cos(n\pi/2), \sin(n\pi/2), \cos(n\pi/4), \sin(n\pi/4)$, etc. in the answer.

Problem 3 By using separation of variables, find a solution for the following partial differential equation with Dirichlet boundary conditions:

$$\begin{aligned} u_t(x,t) + 2u(x,t) &= c^2 u_{xx}(x,t), && x \in [0,\pi], \quad t > 0, \\ u(0,t) &= u(\pi,t) = 0, && t > 0, \\ u(x,0) &= \sin(4x) + 3\sin(x), && x \in [0,\pi], \end{aligned}$$

where c is a positive constant. Explain the details.

Problem 4 Consider the initial value problem for the wave equation on the real line,

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), && -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), && -\infty < x < \infty, \\ u_t(x,0) &= 0, && -\infty < x < \infty, \end{aligned}$$

where c is a positive constant, and $f(x)$ is a given function which has a Fourier transform $\widehat{f}(w)$. Find the Fourier transform of the solution

$$\widehat{u}(w,t) = \int_{-\infty}^{\infty} u(x,t) e^{-iwx} dx.$$

Then using Parseval's identity show the following:

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \, \mathrm{d}x \leq \int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x,$$

for any time $t > 0$. Explain the details.

Problem 5

Show that the following function $u(x, y)$ is harmonic

$$u(x, y) = e^{-x}(x \sin(y) - y \cos(y)).$$

Find a real valued function $v(x, y)$, such that $f(x, y) = u(x, y) + iv(x, y)$ is analytic in $\mathbb{C}$.

Problem 6 Find all Taylor and Laurent series of

$$f(z) = \frac{7}{z^2 - z - 12}$$

around $z = 0$. State the domains where the series converge.

Problem 7 Find all singular points of the following function

$$f(z) = \frac{\sin(z)}{z^3(z-\pi)}$$

and calculate residues at these points.

Hint: you can use the following without proof:

$$\lim_{z\to 0} \frac{z\cos(z) - \sin(z)}{z^2(z-\pi)} = 0.$$

Problem 8 Calculate the integral

$$\oint_{C_r} \frac{z}{2+z^4}\,\mathrm{d}z$$

for r large enough, where the closed path C_r, a quarter circle, is given below. Using this, calculate the following real integral

$$\int_0^\infty \frac{x}{2+x^4}\,\mathrm{d}x.$$