Problem 1 Find the solution $y(t)$ of the initial value problem

$$\begin{cases} \dfrac{d^3y}{dt^3} + 4\dfrac{dy}{dt} = \delta(t+100), \quad t > 0, \\ \\ y(0) = 0 = y'(0), \quad y''(0) = 5, \end{cases}$$

where $\delta(t)$ is the delta function.

[**Solution**]

OBS: Here the impuls/center of the δ is at $t = -100$, and hence $\delta(t+100) = 0$ for all $t > 0$. This was an oversight form our side, the problem was originally intended with $\delta(t-100)$ which would be more standard. We will therefore accept two solutions: (i) Using that $\delta(t+100) = 0$ for all $t > 0$, and (ii) the solution you get using $\delta(t-100)$ instead of $\delta(t+100)$. We give the details in the second case as the first case is similar but simpler:

Setting $Y(s) = \mathcal{L}[y](s)$ and Laplace transforming

$$\frac{d^3y}{dt^3}(t) + 4\frac{dy}{dt}(t) = \delta(t-100), \quad y(0) = 0 = y'(0), \quad y''(0) = 5,$$

yields

$$\begin{aligned} &(s^3Y(s) - y''(0) - sy'(0) - s^2y(0)) + 4\,(sY(s) - y(0)) = \mathrm{e}^{-100s} \\ \Leftrightarrow \qquad &(s^3 + 4s)Y(s) = 5 + \mathrm{e}^{-100s} \\ \Leftrightarrow \qquad &Y(s) = \frac{1}{s^3+4s}\left(5 + \mathrm{e}^{-100s}\right) = 5F(s) + \mathrm{e}^{-100s}F(s), \end{aligned}$$

with

$$F(s) = \frac{1}{s^3+4s}.$$

We expand in partial fractions:

$$\begin{aligned} &\frac{1}{s^3+4s} = \frac{1}{s\,(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{A\,(s^2+4) + Bs^2 + Cs}{s\,(s^2+4)} \\ &\Rightarrow \ A + B = 0, \quad C = 0, \quad 4A = 1 \\ &\Rightarrow \ A = \frac{1}{4} = -B, \quad C = 0 \end{aligned}$$

Taking the inverse Laplace transform,

$$\mathcal{L}^{-1}\left[F(s)\right] = L^{-1}\left[\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right] = \frac{1}{4}(1-\cos 2t) =: f(t),$$

and by the t-shifting rule,

$$\mathcal{L}^{-1}\left[\mathrm{e}^{-100s}F(s)\right] = f(t-100)u(t-100).$$

Finally, we obtain

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \frac{5}{4}(1-\cos 2t) + \frac{1}{4}\Big[1-\cos 2(t-100)\Big]u(t-100).$$

Problem 2 Let the function $f(x)$ be defined on $[-1,1]$ by

$$f(x) = \begin{cases} 1, & -1 \le x \le -\frac{1}{2}, \\ 0, & -\frac{1}{2} < x < \frac{1}{2}, \\ 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

Show that the Fourier series of $f(x)$ is given by

$$S_f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\cos((2n+1)\pi x)}{2n+1}.$$

Sketch the sum $S_f(x)$ on the interval $[-2,2]$ and determine the sum of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

[**Solution**] The function f is defined on $[-1,1]$, therefore $p = 2L = 2$. Since f is even and $L = 1$, we have

$$S_f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n\pi x) + b_n\sin(n\pi x)\right), \text{ where}$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\cdot 2\int_0^1 f(x)dx = \frac{1}{2},$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx = 2\int_0^1 f(x)\cos(n\pi x)dx = 2\int_{\frac{1}{2}}^{1}\cos(n\pi x)dx$$

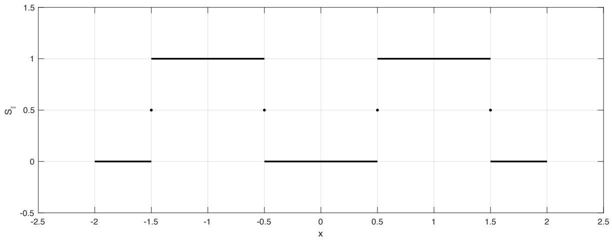
$$= \frac{2}{n\pi}\Big(\sin(n\pi) - \sin\frac{n\pi}{2}\Big) = -\frac{2}{n\pi}\sin\frac{n\pi}{2}, \quad \sin\tfrac{n\pi}{2} = \begin{cases} 0, & n = 2m, \\ (-1)^m, & n = 2m+1, \end{cases}$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)dx = 0.$$

Hence

$$S_f(x) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m+1)} \frac{2}{\pi} \cos((2m+1)\pi x).$$

Sketch of $S_f(x)$:



Since f is continuous and differentiable at $x = 0$,

$$0 = f(0) = S_f(0) = \frac{1}{2} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} \cos 0 \quad \Rightarrow \quad \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} = \frac{\pi}{4}.$$

Problem 3 Find all solutions $z \in \mathbb{C}$ of

$$\sin z = i.$$

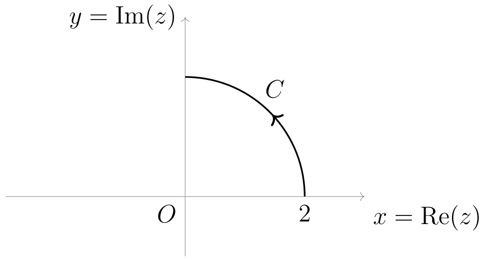
[**Solution**] We multiply by $2ie^{iz}$, solve a quadratic equation, and compute the log:

$$\begin{aligned}
& \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = i \\
\Rightarrow \quad & \left(e^{iz}\right)^2 + 2e^{iz} - 1 = 0 \\
\Rightarrow \quad & e^{iz} = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2} \\
\Rightarrow \quad & iz = \begin{cases} \ln|-1-\sqrt{2}| + i(\operatorname{Arg}(-1-\sqrt{2}) + 2\pi n), \\ \ln|-1+\sqrt{2}| + i(\operatorname{Arg}(-1+\sqrt{2}) + 2\pi n), \end{cases} \qquad n = 0, 1, \ldots
\end{aligned}$$

Since $\operatorname{Arg}(-1-\sqrt{2}) = \pi$, $\operatorname{Arg}(-1+\sqrt{2}) = 0$, we obtain

$$z = \begin{cases} (2n+1)\pi - i\ln(1+\sqrt{2}), \\ 2n\pi - i\ln(\sqrt{2}-1), \end{cases} \qquad n = 0, 1, \ldots .$$

Problem 4 Let C be the part of the circle $|z| = 2$ lying in the first quadrant oriented counter clockwisely.



Calculate the following integrals:

$$(i)\ \int_C z\,dz, \qquad (ii)\ \int_C \bar{z}\,dz, \qquad \text{and} \qquad (iii)\ \int_C \cosh(z)\,dz.$$

[Solution] Note that $C: z(t) = 2e^{it}, t \in \left[0, \frac{\pi}{2}\right]$.

(i) Since z is analytic in $\mathbb{C}$ and $\left(\dfrac{1}{2}z^2\right)' = z$,

$$\int_C z\,dz = \int_{z(0)}^{z(\frac{\pi}{2})} z\,dz = \left[\frac{1}{2}z^2\right]_{z(0)}^{z(\frac{\pi}{2})} = \frac{1}{2}(2i)^2 - \frac{1}{2}2^2 = -4.$$

(ii) By the parametrization,

$$\int_c \bar{z}\,dz = \int_0^{\frac{\pi}{2}} \bar{z}(t)\cdot\dot{z}(t)\,dt = \int_0^{\frac{\pi}{2}} 2e^{-it}\cdot 2ie^{it}\,dt = 4i\int_0^{\frac{\pi}{2}} dt = 2\pi i.$$

(iii) Since $\cosh z$ is analytic in $\mathbb{C}$, $(\sinh z)' = \cosh z$, $\sinh(2i) = \frac{1}{2}(e^{2i} - e^{-2i})$,

$$\int_C \cosh(z)dz = \int_{z(0)}^{z(\frac{\pi}{2})} \cosh z\,dz = \sinh(2i) - \sinh(2) = -\sinh 2 + i\sin 2.$$

Problem 5 Let the function $f(z)$ be defined by

$$f(z) = \frac{1}{1-z} + \frac{1}{2-z} + \frac{1}{2022-z}.$$

Find all Taylor/Laurent series centered at $z_0 = 0$ that represent $f(z)$ for $|z| < 2$. Determine their regions of convergence.

[**Solution**] By results for geometric series and substitution:

$$\frac{1}{1-z} = \sum z^n, \quad |z| < 1,$$

$$\frac{1}{1-z} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} \underset{u=\frac{1}{z}}{=} -u\frac{1}{1-u} \underset{|u|<1}{=} -u\sum_{n=0}^{\infty} u^n \underset{u=\frac{1}{z}}{=} -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}, \quad |u| < 1 \Leftrightarrow |z| > 1,$$

$$\frac{1}{2-z} = \frac{1}{2}\frac{1}{1-\frac{z}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \quad \left|\frac{z}{2}\right| < 1 \Leftrightarrow |z| < 2,$$

$$\frac{1}{2022-z} = \frac{1}{2022}\frac{1}{1-\frac{z}{2022}} = \frac{1}{2022}\sum_{n=0}^{\infty}\left(\frac{z}{2022}\right)^n, \quad \left|\frac{z}{2022}\right| < 1 \Leftrightarrow |z| < 2022.$$

For $|z| < 1$, f is represented by the Taylor series

$$\begin{aligned} f(z) &= \sum z^n + \frac{1}{2}\sum\left(\frac{z}{2}\right)^n + \frac{1}{2022}\sum\left(\frac{z}{2022}\right)^n \\ &= \sum_{n=0}^{\infty}\left(1 + \frac{1}{2^{n+1}} + \frac{1}{2022^{n+1}}\right)z^n. \end{aligned}$$

For $1 < |z| < 2$, f is represented by the Laurent series

$$\begin{aligned} f(z) &= -\sum\frac{1}{z^{n+1}} + \frac{1}{2}\sum\left(\frac{z}{2}\right)^n + \frac{1}{2022}\sum\left(\frac{z}{2022}\right)^n \\ &= -\sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} + \frac{1}{2022^{n+1}}\right)z^n. \end{aligned}$$

Problem 6 The motion of damped soundwaves in an open-ended pipe can be modelled by the following (scaled) initial-boundary value problem for a damped wave equation on the interval $[0, 1]$,

(1) $\quad u_{tt} + 2u_t = u_{xx}, \qquad x \in (0, 1), \quad t > 0,$

(2) $\quad u_x(0, t) = 0 = u_x(1, t), \qquad t > 0,$

(3) $\quad u(x, 0) = 2 - \cos(4\pi x), \qquad x \in [0, 1],$

(4) $\quad u_t(x, 0) = 0, \qquad x \in [0, 1].$

a) Let $\tilde{u}(x, t) = X(x)T(t)$ be a solution of (1) and (2). (i) Find the differential equations and boundary conditions satisfied by X and T, (ii) check that

$$X_n(x) = \cos(n\pi x)$$

is a solution for $n = 0, 1, 2, 3, \ldots$, and (iii) find the solution T_n corresponding to X_n for $n = 0, 1, 2, 3, \ldots$.

Hint: The general solution of 2nd order differential equations can be found in the attachment.

b) Find the solution $u(x, t)$ of the problem (1), (2), (3), and (4).

[**Solution**]

(a) $\tilde{u}(x, t) = X(x)T(t)$ solves (1) and (2).

$$(1) \;\Rightarrow X(x)T''(t) + 2X(x)T'(t) = X''(x)T(t)$$

Dividing by $X(x)T(t) \neq 0$, we have

$$\frac{T'' + 2T'}{T}(t) = \frac{X''}{X}(x) = \text{ const } = k$$
$$\Rightarrow T''(t) + 2T'(t) = kT(t), \quad X''(x) = kX(x).$$

By the same token:

$$(2) \Rightarrow X'(0)T(t) = 0 = X'(1)T(t)$$
$$\overset{T(t)\neq 0}{\Rightarrow} \quad X'(0) = 0 = X'(1).$$

We obtain the following equations for $X(x)$ and $T(t)$:

(5) $X''(x) = kX(x), \quad X'(0) = 0 = X'(1),$

(6) $T''(t) + 2T'(t) = kT(t).$

We check that $X_n(x) = \cos(n\pi x)$ solves (5):

$$X_n''(x) = ((-n\pi)\sin(n\pi x))' = -(n\pi)^2\cos(n\pi x) = -(n\pi)^2 X_n(x),$$
$$X_n'(0) = -\,n\pi\sin(n\pi x)|_{x=0} = 0,$$
$$X_n'(1) = -n\pi\sin(n\pi) = 0,$$

i.e. $X_n(x)$ solves (5) with $k = -(n\pi)^2$.

Let us find the solution of (6) with $k = -(n\pi)^2$:

$$(6) \;\Rightarrow\; T_n''(t) + 2T_n'(t) + (n\pi)^2 T_n(t) = 0.$$

Characteristic equation: $r_n^2 + 2r_n + (n\pi)^2 = 0$

$$\Rightarrow r_n = \frac{-2 \pm \sqrt{4 - 4(n\pi)^2}}{2} = -1 \pm \sqrt{1 - (n\pi)^2}$$

$$\Rightarrow \begin{cases} r_0 & = -1 \pm 1 = -2 \text{ or } 0 \\ r_n & = -1 \pm i\omega_n, \quad \omega_n = \sqrt{(n\pi)^2 - 1}, \;\; n \geqslant 1. \end{cases}$$

Hence

$$T_0(t) = A_0 e^{-2t} + B_0,$$
$$T_n(t) = e^{-t}\left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right).$$

(b) Candidate solution: Let $N \in \mathbb{N}$ be large and define

$$u(x,t) = \sum_{n=0}^{N} u_n(x,t) = \sum_{n=0}^{N} X_n(x) T_n(t).$$

By a) $u_n(x,t)$ solves (1) and (2), and then so does $u(x,t)$ by superposition ((1), (2) linear and homogeneous). Determine coefficients A_n, B_n so that (3), (4) is satisfied:

$$(3) \quad \Rightarrow \; 2 - \cos(4\pi x) = u(x,0) = \sum_{n=0}^{N} X_n(x) T_n(0) = (A_0 + B_0) + \sum_{n=1}^{N} A_n \cos(n\pi x)$$
$$\Rightarrow \; A_0 + B_0 = 2, \quad A_4 = -1, \quad A_n = 0 \text{ otherwise.}$$

We compute $\partial_t u$:

$$u_t(x,t) = -\,2e^{-t}A_0 + \sum_{n=1}^{N} (-e^{-t}) T_n(t) X_n(x)$$
$$+ \sum_{n=1}^{N} e^{-t}\Big(A_n\,(-\omega_n)\sin(\omega_n t) + B_n \omega_n \cos(\omega_n t)\Big) X_n(x),$$

and observe that

$$(4) \quad \Rightarrow \quad 0 = u_t\,(x,0) = -A_0 - \sum_{n=1}^{N} A_n \cos(n\pi x) + \sum_{n=1}^{N} B_n \omega_n \cos(n\pi x)$$

$$\Rightarrow \quad A_0 = 0, \quad B_n \omega_n - A_n = 0$$

In conclusion (take $N \geq 4$), the solution of (1)-(4) is

$$u(x,t) = \sum_{n=1}^{N} u_n(x,t) = 2 - e^{-t}\left(\cos(\omega_4 t) + \frac{\sin(\omega_4 t)}{\omega_4}\right)\cos(4\pi x).$$

Problem 7 Show that if $g(z)$ and $h(z)$ are analytic functions, $g(0) \neq 0 \neq h(0)$, and $h'(0) = 0$, then

$$\operatorname*{Res}_{z=0}\left[\frac{g(z)}{z^2 h(z)}\right] = \frac{g'(0)}{h(0)}.$$

Compute the integral

$$\oint_{|z-i|=2} \frac{e^{-i\pi z}}{z\sin(\pi z)}\,dz.$$

Hint: You can use without proof that $\operatorname{sinc}(z)$ is an analytic function such that $\operatorname{sinc}(0) = 1$, $\operatorname{sinc}'(0) = 0$, and $z\operatorname{sinc}(z) = \sin z$ for $z \in \mathbb{C}$.

[**Solution**] Note that

$$f(z) := \frac{g(z)}{z^2 h(z)}$$

has an order 2 pole at $z = 0$, because $g(0) \neq 0 \neq h(0)$ and $z^2 h(z)$ has a zero of oder 2 at $z = 0$. Since $h'(0) = 0, h(0) \neq 0$, we find that

$$\mathrm{Res}_{z=0}\, f(z) = \lim_{z\to 0}\left(z^2 f(z)\right)' = \lim_{z\to 0}\left(\frac{g(z)}{h(z)}\right)' = \lim_{z\to 0}\left(\frac{g'(z)}{h(z)} - \frac{g(z)}{h^2(z)}h'(z)\right) = \frac{g'(0)}{h(0)}.$$

[Alternative: By the residue theorem and Cauchy's integral formula for the derivative

$$\operatorname*{Res}_{z=0} f(z)\,dz = \frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{1}{z^2}\frac{g(z)}{h(z)}\,dz = \frac{d}{dz}\left(\frac{g(z)}{h(z)}\right)_{z=0} = \frac{g'(0)}{h(0)},$$

where C is any simple closed curve encircling only one singularity, at $z = 0$.]

Integral:

$$I = \oint_{|z-i|=2} f(z)dz$$

where

$$f(z) = \frac{e^{i\pi z}}{z\sin(\pi z)} = \frac{g(z)}{\tilde{h}(z)} = \frac{g(z)}{z^2 h(z)} \quad \text{for} \quad \tilde{h}(z) = z\sin(\pi z), \quad h(z) = \pi\operatorname{sinc}(\pi z).$$

(i) Singularities of $f(z)$: $\sin(\pi z)$ has order 1 zeros at $z = n \in \mathbb{Z}$, and $z\sin(\pi z)$ has an order 2 zero at $z = 0$ and order 1 zeros otherwise. Since $e^{i\pi z} \neq 0$, $f(z)$ has an order 2 pole at $z = 0$, order 1 poles at $z \in \mathbb{Z}$, $z \neq 0$.

(ii) Singularities within $|z - i| = 2$:

$$z = -1, \quad z = 0, \quad z = 1.$$

(iii) Residues at 1st order poles:

$$\operatorname*{Res}_{z=\pm 1} f(z) = \frac{g(z)}{\tilde{h}'(z)}\bigg|_{z=\pm 1} = \frac{e^{i\pi z}}{(z\sin(\pi z))'}\bigg|_{z=\pm 1} = \frac{e^{\pm i\pi}}{(\sin(\pi z) + \pi z\cos(\pi z))_{z=\pm 1}}$$
$$= \frac{-1}{0 \pm \pi\cos(\pm\pi)} = \frac{-1}{\pm\pi} = \pm\frac{1}{\pi}.$$

Residue at 2nd order pole:

$$\operatorname*{Res}_{z=0} f(z) = \frac{g'(0)}{h(0)} = \frac{-i\pi e^0}{\pi \operatorname{sinc} 0} = -i.$$

(iv) By the residue Theorem:

$$I = 2\pi i\left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-1} f(z)\right) = 2\pi i\left(\frac{1}{\pi} - i - \frac{1}{\pi}\right) = 2\pi.$$