Problem 1 Find a solution $y(t)$ for the following equation for $t > 0$ by using the Laplace transform:

$$y'(t) + y(t) = f(t), \quad f(t) = \begin{cases} 0, & 0 < t < 2, \\ t, & 2 < t, \end{cases} \qquad y(0) = 2.$$

[**Solution**] By applying Laplace transform to the equation, we obtain

$$\begin{aligned} sY(s) - y(0) + Y(s) &= \int_0^\infty f(t)e^{-st}\,\mathrm{d}t = \int_2^\infty te^{-st}\,\mathrm{d}t \\ &= -\left[\frac{te^{-st}}{s}\right]_2^\infty + \frac{1}{s}\int_2^\infty e^{-st}\,\mathrm{d}t = \frac{e^{-2s}(2s+1)}{s^2}. \end{aligned}$$

Hence,

$$Y(s) = \frac{e^{-2s}}{(s+1)s^2} + \frac{2e^{-2s}}{(s+1)s} + \frac{2}{s+1}.$$

Partial fractional decomposition gives us

$$\frac{1}{(s+1)s^2} = \frac{A}{s+1} + \frac{Bs+C}{s^2} \implies A+B = 0, B+C = 0, C = 1 \implies A = C = 1, B = -1.$$

Thus

$$\begin{aligned} Y(s) &= e^{-2s}\left(\frac{1}{s+1} + \frac{1}{s^2} - \frac{1}{s}\right) + 2e^{-2s}\left(\frac{1}{s} - \frac{1}{s+1}\right) + \frac{2}{s+1} \\ &= e^{-2s}\left(\frac{1}{s} - \frac{1}{s+1} + \frac{1}{s^2}\right) + \frac{2}{s+1}. \end{aligned}$$

With the help of t-shifting, we find the inverse Laplace transform of $Y(s)$:

$$y(t) = 2e^{-t} + u(t-2)\left(-e^{-(t-2)} + t - 1\right).$$

Problem 2 Let $f(x)$ be a π-periodic function on $\mathbb{R}$, defined by

$$f(x) = \max\left\{0, x - \frac{\pi}{2}\right\}, \quad \text{for } 0 < x < \pi.$$

(i) Sketch this Fourier series on the interval $-\pi < x < \pi$. (ii) Find the Fourier series of $f(x)$. (iii) What is the value of this Fourier series at $x = \pi$?

Hint: you may use $\cos(n\pi/2), \sin(n\pi/2), \cos(n\pi/4), \sin(n\pi/4)$, etc. in the answer.

[**Solution**] Since the period is $2L = \pi$, we have the Fourier series representation

$$S_f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(2kx) + b_k \sin(2kx) \right),$$

where the coefficients can be calculated in the following:

$$a_0 = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(x)\,\mathrm{d}x = \frac{1}{\pi} \int_{-\pi/2}^{0} \left(x + \frac{\pi}{2} \right) \mathrm{d}x = \frac{\pi}{8};$$

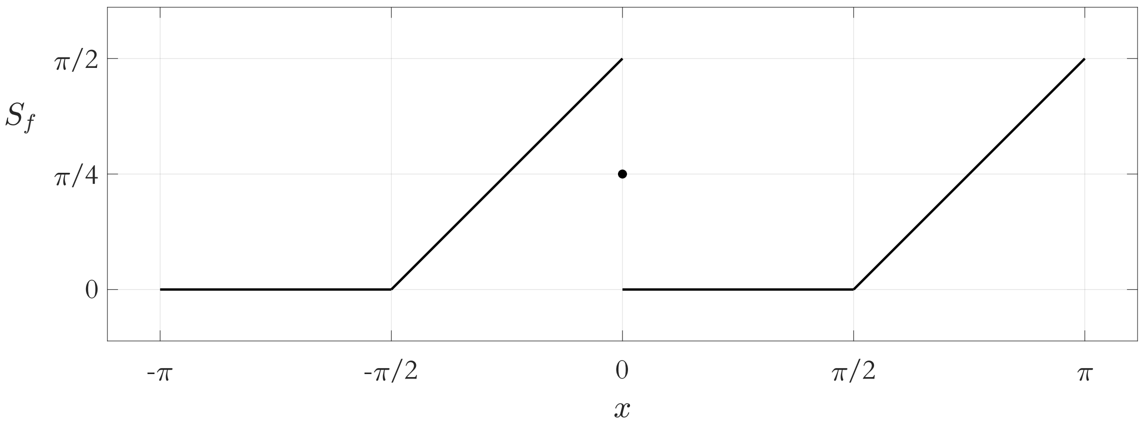
$$\begin{aligned} a_k &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \cos(2kx)\,\mathrm{d}x = \frac{2}{\pi} \int_{-\pi/2}^{0} \left(x + \frac{\pi}{2} \right) \cos(2kx)\,\mathrm{d}x \\ &= \frac{2}{\pi} \left[\frac{1-(-1)^k}{(2k)^2} \right] = \frac{1-(-1)^k}{2\pi k^2}; \end{aligned}$$

and

$$\begin{aligned} b_k &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \sin(2kx)\,\mathrm{d}x = \frac{2}{\pi} \int_{-\pi/2}^{0} \left(x + \frac{\pi}{2} \right) \sin(2kx)\,\mathrm{d}x \\ &= -\frac{1}{2k}. \end{aligned}$$

(NB: for calculating integrals, one can use integration by parts, or the formula from the table.) The Fourier series is discontinuous at $x = \pi$, and thus

$$S_f(\pi) = \frac{\lim_{x\to\pi-} S_f(x) + \lim_{x\to\pi+} S_f(x)}{2} = \frac{\pi/2 + 0}{2} = \frac{\pi}{4}.$$



Problem 3 By using separation of variables, find a solution for the following partial differential equation with Dirichlet boundary conditions:

$$\begin{aligned} u_t(x,t) + 2u(x,t) &= c^2 u_{xx}(x,t), && x \in [0,\pi], \quad t > 0, \\ u(0,t) &= u(\pi,t) = 0, && t > 0, \\ u(x,0) &= \sin(4x) + 3\sin(x), && x \in [0,\pi], \end{aligned}$$

where c is a positive constant. Explain the details.

[**Solution**] Assume a solution u is of the form $u(x,t) = F(x)G(t)$. Then the PDE implies

$$F(x)G'(t) + 2F(x)G(t) = c^2 F''(x)G(t).$$

We seek for non-zero solutions, and hence assume F and G are non-zero functions. We obtain

$$\frac{G'(t) + 2G(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}.$$

The left hand side only depends on t and the right hand side only depends on x, thus we must have

$$\frac{G'(t) + 2G(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k,$$

for some constant k. First we consider the ODE $F'' = kF/c^2$. From the boundary condition, we have $F(0)G(t) = F(\pi)G(t) = 0$ for all t. Since G is not zero function, we obtain $F(0) = F(\pi) = 0$. If $k > 0$ then $F(x) = Ae^{\sqrt{k/c^2}x} + Be^{-\sqrt{k/c^2}x}$ but the boundary condition gives us $A = B = 0$. This means $u = 0$. If $k = 0$ then $F(x) = Ax + B$ but again, the boundary condition tells us $A = B = 0$, which means we only get the zero solution. However, if we have $0 > k := -p^2$, we find non-zero solution F such that

$$F(x) = A\cos\left(\frac{p}{c}x\right) + B\sin\left(\frac{p}{c}x\right), F(\pi) = F(0) = 0 \implies A = 0, \frac{p}{c} = n \in \mathbb{Z}.$$

Therefore, $F_n(x) = \sin(nx)$ satisfies the ODE with the boundary condition. For each F_n we want to find G_n that satisfies

$$\frac{G_n'(t)}{G_n(t)} + 2 = k = -(cn)^2.$$

The general solution is $G_n = B_n e^{-(c^2n^2+2)t}$. Now, by using the superposition of F_nG_n,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-(c^2n^2+2)t},$$

satisfies the original linear homogeneous PDE. We want to find the solution which satisfies the initial condition $u(x,0) = \sin(4x) + 3\sin(x)$. This gives us the coefficients $B_1 = 3, B_4 = 1$, and $B_n = 0$ $(n \neq 1, 4)$. Thus, as a conclusion, the answer for the given problem is

$$u(x,t) = 3\sin(x)e^{-(c^2+2)t} + \sin(4x)e^{-(16c^2+2)t}.$$

Problem 4 Consider the initial value problem for the wave equation on the real line,

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), && -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), && -\infty < x < \infty, \\ u_t(x,0) &= 0, && -\infty < x < \infty, \end{aligned}$$

where c is a positive constant, and $f(x)$ is a given function which has a Fourier transform $\widehat{f}(w)$. Find the Fourier transform of the solution

$$\widehat{u}(w,t) = \int_{-\infty}^{\infty} u(x,t)e^{-iwx}dx.$$

Then using Parseval's identity show the following:

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \,\mathrm{d}x \le \int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x,$$

for any time $t > 0$. Explain the details.

[**Solution**] By applying Fourier transform for the PDE, we have

$$\widehat{u}_{tt}(w,t) = -c^2 w^2 \widehat{u}(w,t).$$

With the initial conditions $\widehat{u}_t(w,0) = 0$ and $\widehat{u}(w,0) = \widehat{f}(w)$, we obtain

$$\widehat{u}(w,t) = A(w)e^{itcw} + B(w)e^{-itcw} = \frac{1}{2}\widehat{f}(w)(e^{-ictw} + e^{ictw}).$$

By using the Parseval's idendity, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x,t)|^2 \,\mathrm{d}x &= \int_{-\infty}^{\infty} |\widehat{u}(w,t)|^2 \,\mathrm{d}w = \int_{-\infty}^{\infty} |\frac{1}{2}\widehat{f}(w)(e^{-ictw} + e^{ictw})|^2 \,\mathrm{d}w \\ &= \int_{-\infty}^{\infty} |\widehat{f}(w,t)\cos(ctw)|^2 \,\mathrm{d}w. \end{aligned}$$

Since c, w and t are real numbers, we obtain

$$\int_{-\infty}^{\infty} |\widehat{f}(w,t)\cos(ctw)|^2 \,\mathrm{d}w \le \int_{-\infty}^{\infty} |\widehat{f}(w,t)|^2 \,\mathrm{d}w = \int_{-\infty}^{\infty} |f(x,t)|^2 \,\mathrm{d}x,$$

where in the last inequality we used the Parseval's identity again.

Problem 5

Show that the following function $u(x, y)$ is harmonic

$$u(x, y) = e^{-x}(x \sin(y) - y \cos(y)).$$

Find a real valued function $v(x, y)$, such that $f(x, y) = u(x, y) + iv(x, y)$ is analytic in $\mathbb{C}$.

[**Solution**] We have

$$\begin{aligned} u_x &= \mathrm{e}^{-x}\sin(y) - x\mathrm{e}^{-x}\sin(y) + y\mathrm{e}^{-x}\cos(y), \\ u_{xx} &= -2\mathrm{e}^{-x}\sin(y) + x\mathrm{e}^{-x}\sin(y) - y\mathrm{e}^{-x}\cos(y), \\ u_y &= x\mathrm{e}^{-x}\cos(y) + y\mathrm{e}^{-x}\sin(y) - \mathrm{e}^{-x}\cos(y), \\ u_{yy} &= -x\mathrm{e}^{-x}\sin(y) + 2\mathrm{e}^{-x}\sin(y) + y\mathrm{e}^{-x}\cos(y), \end{aligned}$$

Thus $u_{xx} + u_{yy} = 0$ and u is a harmonic function. Now, from the Cauchy-Riemann equation, $v_x = u_y$ and $v_x = -u_y$, we obtain

$$v(x, y) = \int v_x \, \mathrm{d}x = -\int u_y \, \mathrm{d}x = \mathrm{e}^{-x}(x\cos(y) + y\sin(y)) + C(y),$$

where $C(y)$ is an integration constant depending on y. Now, $v_y = u_x$ should hold. Thus $C_y(y) = 0$ and hence $C(y) = const$. Therefore,

$$v(x, y) = \mathrm{e}^{-x}(x\cos(y) + y\sin(y)) + C,$$

where C is a constant.

Problem 6 Find all Taylor and Laurent series of

$$f(z) = \frac{7}{z^2 - z - 12}$$

around $z = 0$. State the domains where the series converge.

[**Solution**] Partial fractional decomposition gives us

$$\frac{7}{z^2 - z - 12} = \left(\frac{1}{z-4} - \frac{1}{z+3}\right).$$

For the first term, we have

$$\frac{1}{z-4} = \frac{1}{4(z/4 - 1)} = -\sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{z}{4}\right)^k$$

for $|z| < 4$ and

$$\frac{1}{z-4} = \frac{1/z}{1-4/z} = \sum_{k=0}^{\infty} \frac{1}{z}\left(\frac{4}{z}\right)^k = \sum_{k=0}^{\infty}\left(\frac{4^k}{z^{k+1}}\right)$$

for $|z| > 4$. Similarly, we have

$$\frac{1}{z+3} = \frac{1}{z-(-3)} = \begin{cases} \sum_{k=0}^{\infty} \frac{1}{3}\left(-\frac{z}{3}\right)^k & \text{for } |z| < 3 \\ \sum_{k=0}^{\infty} \left(\frac{(-3)^k}{z^{k+1}}\right) & \text{for } |z| > 3. \end{cases}$$

As a conclusion, we have

$$f(z) = \begin{cases} \sum_{k=0}^{\infty}\left(-\frac{1}{4^{k+1}} + \frac{1}{3}\left(-\frac{1}{3}\right)^k\right) z^k & \text{for } |z| < 3 \\ \sum_{k=0}^{\infty}\left(-\frac{1}{4^{k+1}}\right) z^k + \sum_{k=1}^{\infty}\left(\frac{(-3)^{k-1}}{z^k}\right) & \text{for } 3 < |z| < 4 \\ \sum_{k=1}^{\infty}\left((-3)^{k-1} + 4^{k-1}\right)\frac{1}{z^k} & \text{for } 4 < |z|. \end{cases}$$

Problem 7 Find all singular points of the following function

$$f(z) = \frac{\sin(z)}{z^3(z-\pi)}$$

and calculate residues at these points.

Hint: you can use the following without proof:

$$\lim_{z\to 0} \frac{z\cos(z) - \sin(z)}{z^2(z-\pi)} = 0.$$

[**Solution**] The function $f(z)$ has two singular points: $z = 0$ and π. Notice that $\sin(z)$ has zeros at $n\pi$ for any integer n.

First, $z = \pi$ is a removable singularity. In fact, defining value $f(\pi) = 1/\pi^3$ would make $f(z)$ continuous at $z = \pi$. Therefore, the residue at this point is 0.

On the other hand, $z = 0$ is the second order pole and

$$\begin{aligned} \mathrm{Res}_{z=0}[f(z)] &= \lim_{z\to 0}(z^2 f(z))' = \lim_{z\to 0}\left(\frac{\sin(z)}{z(z-\pi)}\right)' \\ &= \lim_{z\to 0} \frac{z\cos(z)-\sin(z)}{z^2(z-\pi)} - \frac{\sin(z)}{z(z-\pi)^2} = -\frac{1}{\pi^2}, \end{aligned}$$

where in the last line we used

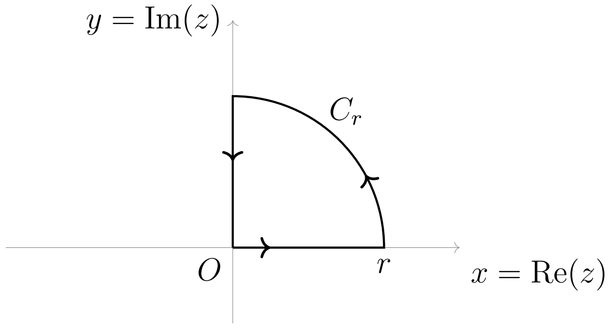
$$\lim_{z\to 0} \frac{z\cos(z) - \sin(z)}{z^2(z-\pi)} = 0.$$

Problem 8 Calculate the integral

$$\oint_{C_r} \frac{z}{2+z^4}\,\mathrm{d}z$$

for r large enough, where the closed path C_r, a quarter circle, is given below. Using this, calculate the following real integral

$$\int_0^\infty \frac{x}{2+x^4}\,\mathrm{d}x.$$



[**Solution**] The function $f(z)$ has its first order poles at $z_k := 2^{1/4}e^{\pi i(k/2+1/4)}$ for $k = 0, 1, 2, 3$. For r large enough, C_r encloses only the pole for $k = 0$. Thus, using the residue theorem, we have

$$\oint_{C_r} \frac{z}{2+z^4}\,\mathrm{d}z = 2\pi i \mathrm{Res}_{z=z_0}[f(z)].$$

Note that

$$\mathrm{Res}_{z=z_0}[f(z)] = \left.\frac{z}{(2+z^4)'}\right|_{z=z_0} = -\frac{i}{4\sqrt{2}}.$$

Hence the path integral is $\oint_{C_r} \frac{z}{2+z^4}\,\mathrm{d}z = \frac{\pi}{2\sqrt{2}}$.

Now, let $I_r := \int_0^r \frac{x}{2+x^4}\,\mathrm{d}x$. Then

$$\oint_{C_r} \frac{z}{2+z^4}\,\mathrm{d}z = I_r + \int_{S_r} \frac{z}{2+z^4}\,\mathrm{d}z + \int_{ri}^0 \frac{z}{2+z^4}\,\mathrm{d}z$$

where the curve $S_r : z(t) = re^{it}$, $t \in [0, \pi/2]$. Notice that the third term is same as I_r by changing variables $z = ix$:

$$\int_{ri}^{0} \frac{z}{2+z^4}\,\mathrm{d}z = -\int_{r}^{0} \frac{x}{2+x^4}\,\mathrm{d}x = I_r.$$

Using the ML inequality, we have

$$\left|\int_{S_r} \frac{z}{2+z^4}\,\mathrm{d}z\right| \leq \max_{z\in S_r} |f(z)| 2\pi r \leq \frac{r}{r^4-2} 2\pi r \to 0, \mathrm{as}\ r \to \infty.$$

Therefore, we conclude

$$\int_{0}^{\infty} \frac{x}{2+x^4}\,\mathrm{d}x = 2 \lim_{r\to\infty} I_r + 0.$$

This means $\lim_{r\to\infty} I_r = \frac{\pi}{4\sqrt{2}}$.