From Kreyszig (10th), chapter 6.1

1 Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(t) = 2t + 8$. Then it follows that

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st}(2t+8)\,dt = 2\int_0^\infty te^{-st}\,dt + 8\int_0^\infty e^{-st}\,dt = \frac{2}{s^2} + \frac{8}{s}.$$

In the last calculation we have used integration by part to calculate that

$$\int_0^\infty te^{-st} = \frac{1}{s^2}.$$

In the last calculation we have used integration by parts.

12 We have

$$f(t) = \begin{cases} t, & \text{for } 0 < t < 1, \\ 1, & \text{for } 1 < t < 2, \\ 0, & \text{for } t > 2. \end{cases}$$

Therefore

$$\mathcal{L}(f)(s) = \int_0^1 te^{-st}dt + \int_1^2 e^{-st}dt.$$

Using integration by parts for the first integral, we get

$$\begin{aligned} \mathcal{L}(f)(s) &= -\frac{1}{s}e^{-st}t\Big|_0^1 - \int_0^1 \left(-\frac{1}{s}e^{-st}\right)dt - \frac{1}{s}e^{-st}\Big|_1^2 \\ &= -\frac{1}{s}e^{-s} + \frac{1}{s}\left(-\frac{1}{s}e^{-st}\Big|_0^1\right) - \frac{1}{s}e^{-2s} + \frac{1}{s}e^{-s} \\ &= \frac{1}{s^2}\left(1 - e^{-s}\right) - \frac{1}{s}e^{-2s}. \end{aligned}$$

23 Let $\mathcal{L}(f)(t) = F(s)$. By applying the definition of the Laplace transform given in (1) on page 204 to $f(ct)$ we see that

$$\mathcal{L}(f(ct)) = \int_0^\infty e^{-st}f(ct)dt.$$

The substituting $\tau = ct$ we obtain

$$\int_0^\infty e^{-st}f(ct)dt = \int_0^\infty e^{-(s/c)\tau}f(\tau)\frac{d\tau}{c} = \frac{1}{c}F\left(\frac{s}{c}\right).$$

We have thus shown that

$$\mathcal{L}(f(ct)) = \frac{1}{c}F\left(\frac{s}{c}\right). \tag{1}$$

For part two of the exercise we use table 6.9.14 and see that

$$\mathcal{L}(\cos(t)) = \frac{s}{s^2+1}.$$

Then applying the formula (1) from above we see that

$$\begin{aligned} \mathcal{L}(\cos(\omega t)) &= \frac{1}{\omega}\frac{s/\omega}{(s/\omega)^2+1} \\ &= \frac{s}{s^2+\omega^2}. \end{aligned}$$

25 Let

$$F(s) = \frac{0.2s + 1.4}{s^2 + 1.96}.$$

We know that

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2},$$

and that

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

Thus observing that $1.4^2 = 1.96$ it follows that

$$\begin{aligned} \frac{0.2s + 1.4}{s^2 + 1.96} &= 0.2\frac{s}{s^2 + 1.96} + \frac{1.4}{s^2 + 1.96} \\ &= 0.2\mathcal{L}(\cos(1.4t)) + \mathcal{L}(\sin(1.4t)) \\ &= \mathcal{L}(0.2\cos(1.4t) + \sin(1.4t)) \\ \Rightarrow f(t) &= 0.2\cos(1.4t) + \sin(1.4t) \end{aligned}$$

26 We first observe that (6) i tabellen side 207.

$$\frac{5s + 1}{s^2 - 25} = \frac{12}{5}\frac{1}{s + 5} + \frac{13}{5}\frac{1}{s - 5},$$

by using fractions by parts. Then using (6) from page 207 we see that

$$\frac{12}{5}\frac{1}{s + 5} + \frac{13}{5}\frac{1}{s - 5} = \frac{12}{5}\mathcal{L}(e^{-5t}) + \frac{13}{5}\mathcal{L}(e^{5t}).$$

Hence it follows that

$$f(t) = \frac{12}{5}e^{-5t} + \frac{13}{5}e^{5t}.$$

Alternatively one can do as follows:

$$\begin{aligned} \frac{5s + 1}{s^2 - 25} &= 5\frac{s}{s^2 - 25} + \frac{1}{5}\frac{5}{s^2 - 25} \\ &= 5\mathcal{L}(\cosh(5t)) + \frac{1}{5}\mathcal{L}(\sinh(5t)) \\ &= \mathcal{L}(5\ \cosh(5t) + \frac{1}{5}\ \sinh(5t)) \\ \Rightarrow f(t) &= 5\ \cosh(5t) + \frac{1}{5}\ \sinh(5t). \end{aligned}$$

Note that these answers are equivalent.

31 We factorize the polynominal in the denominator and see that

$$F(s) = \frac{-s + 11}{s^2 - 2s - 3} = \frac{-s + 11}{(s + 1)(s - 3)} = \frac{2(s + 1) - 3(s - 3)}{(s - 3)(s + 1)} = 2\frac{1}{s - 3} - 3\frac{1}{s + 1}.$$

Each of the parts of the last expression has a know Laplace transform (table 6.1.6) Thus we see that

$$\mathcal{L}^{-1}(F)(t) = 2e^{3t} - 3e^{-t}.$$

40 Since

$$F(s) = \frac{4}{s^2 - 2s - 3} = \frac{4}{(s - 1)^2 - 4},$$

we may use s-Shifiting, the sinh - transform and linearity of $\mathcal{L}^{-1}$, to find

$$\begin{aligned} e^{-t}f(t) &= \mathcal{L}^{-1}(F(s+1)) \\ &= \mathcal{L}^{-1}(\frac{2\cdot 2}{s^2-2^2}) = 2\sinh 2t. \end{aligned}$$

That is

$$\mathcal{L}^{-1}(F(s)) = f(t) = 2e^t \sinh 2t.$$

A Find the partial fraction decomposition of the following functions. When you have found an answer; check that your solution is correct by adding all the parts of the partial fraction decomposition together again.

1.
$$f(x) = -\frac{x+1}{(2x+1)(x-1)}$$

2.
$$g(x) = \frac{16x-6}{(2x-5)(3x+1)}$$

3.
$$h(x) = \frac{1}{x(x+1)^2}$$

Solution exercise A1.
Since the denominator is a product of two linear factors $(x-a)(x-b)$ with $a \neq b$ we know that there exist constants A and B such that

$$-\frac{x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}.$$

To find the constants A and B we write sum the right hand side of the equation above so we can write it as one fraction;

$$\frac{A}{2x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(2x-1)}{(2x+1)(x-1)} = \frac{(2B+A)x - A + B}{(2x+1)(x-1)}$$

Thus it follows that $2A+B = -1$ and $A+B = -1$. Solving these equations we see that $A = -1/3$ and $B = 2/3$. Then it follows that

$$f(x) = -\frac{x+1}{(2x+1)(x-1)} = \frac{1}{3(2x+1)} - \frac{2}{3(x-1)}.$$

To check that we have not made any mistakes, we sum the fractions and see that we get the expresion we started with:

$$\frac{1}{3(2x+1)} - \frac{2}{3(x-1)} = \frac{x-1-2(2x+1)}{3(2x+1)(x-1)} = \frac{-x-1}{(2x+1)(x-1)}.$$

Solution exercise A2.
We do the same as in the previous exercise and calculate:

$$\frac{16x-6}{(2x-5)(3x+1)} = \frac{A}{2x-5} + \frac{B}{3x+1} = \frac{(3x+1)A + (2x-5)B}{(2x-5)(3x+1)} = \frac{x(3A+2B)+A-5B}{(2x-5)(3x+1)}$$

To find the constants A and B we must solve the equations $3A+2B = 16$ and $A - 5B = 6$. It follows that $A = 4$ and B=2. We can now see that

$$\frac{16x-6}{(2x-5)(3x+1)} = \frac{4}{(2x-5)} + \frac{2}{3x+1}.$$

To check that we have not made any mistakes, we sum the fractions and see that we get the expresion we started with:

$$\frac{4}{(2x-5)} + \frac{2}{3x+1} = \frac{4(3x+1)+2(2x-5)}{(2x-5)(3x+1)} = \frac{16x-6}{(2x-5)(3x+1)}.$$

Solution exercise A3.
We observe that the denumerator consits of repiting linear factors and we must thus use a slightly different approach than in the previous exercises. We know there exists constants A, B and C such that

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

To find the constants A, B and C we calculate the right hand side:

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2 + B(x+1)x + Cx}{x(x+1)^2} = \frac{(A+B)x^2 + (2A+B+C)x + A}{x(x+1)^2}$$

We see that it must be the case that $A+B=0$, $2A+B+C=0$ and $A=1$. Solving this system of equations we see that $A=1$, $B=-1$ and $C=-1$. Thus it follows that

$$\frac{1}{x(x+1)^2} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}.$$

To check that we have not made any mistakes, we sum the fractions and see that we get the expresion we started with:

$$\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} = \frac{(x+1)^2 - x(x+1) - x}{x(x+1)^2} = \frac{x^2+2x+1-x^2-x-x}{x(x+1)^2} = \frac{1}{x(x+1)^2}.$$