Løsningsforslag - Øving 11

Fra Kreyszig (10th), avsnitt 14.3

2 The circle $C: |z-1-i| = \pi/2$ can be written as $|z-(1+i)| = \pi/2$ and corresponds to the circle with center $i+1$ and radius $\pi/2$.

Let

$$f(z) = \frac{z^2}{z^2-1} = \frac{z^2}{(z-1)(z+1)}.$$

$f(z)$ is not analytic at -1 and 1.

The circle C encloses the point $z_0 = 1$, therefore we write

$$\oint_C f(z)dz = \oint_C \frac{z^2}{z^2-1}dz = \oint_C \frac{g(z)}{z-z_0}dz = \oint_C \frac{g(z)}{z-1}dz, \qquad \text{with} \quad g(z) = \frac{z^2}{z+1}.$$

By Cauchy's integral formula we obtain

$$\oint_C f(z)dz = \oint_C \frac{g(z)}{z-1}dz = 2\pi i g(1) = 2\pi i \left[\frac{z^2}{z+1}\right]_{z=1} = 2\pi i \frac{1}{2} = \pi i.$$

3 $C: \quad |z+i| = 1.41$

$$|-1+i| = \sqrt{2} > 1.41$$
$$|1+i| = \sqrt{2} > 1.41$$

$$\implies z=-1 \text{ og } z=1 \text{ på utsiden av } C \implies f(z) \text{ analytisk i } D = \{z : |z+1| \leq 1.41 + \epsilon\}$$

$$\implies \oint_C f(z)dz = 0$$

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$$\oint_C \frac{z+2}{z-2}dz \qquad C: |z-1| = 2 \quad \text{mot urviseren}$$

$$|2-i| = 1 < 2 \implies z = 2 \quad \text{på innsiden av C}$$

$g(z) = z+2$ er analytisk i $\mathbb{C}$

$$\implies \oint_C \frac{z+2}{z-2}dz = \oint_C \frac{g(z)}{z-2}dz = 2\pi i g(2) = 2\pi i 4 = 8\pi i$$

Fra Kreyszig (10th), avsnitt 14.4

2

$$\oint_C \frac{z^6}{(2z-1)^6}dz = \oint_C \frac{z^6}{\left[2\left(z-\frac{1}{2}\right)\right]^6}dz = \oint_C \frac{z^6}{2^6\left(z-\frac{1}{2}\right)^6} = \oint_C \frac{f(z)}{(z-z_0)^6} = \frac{2\pi i}{5!}f^{(5)}(z_0),$$

with

$$f(z) = \frac{z^6}{2^6}, \qquad z_0 = \frac{1}{2}.$$

Notice that $f(z)$ is an entire function (hence it is analytic on and inside the unit circle C) and z_0 lies inside C.

We have

$$f^{(5)}(z_0) = \frac{6!}{2^6}z\Big|_{z=\frac{1}{2}} = \frac{6!}{2^7},$$

hence

$$\oint_C \frac{z^6}{(2z-1)^6}dz = \frac{2\pi i}{5!}\frac{6!}{2^7} = \frac{3\pi i}{2^5} = \frac{3\pi i}{32}.$$

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$$\oint_C \frac{\cos z}{z^{2n+1}}dz = \oint_C \frac{f(z)}{(z-z_0)^{2n+1}} = \frac{2\pi i}{(2n)!}f^{(2n)}(z_0), \qquad n = 0, 1, \ldots$$

with $f(z) = \cos z$ and $z_0 = 0$. Notice that $\cos(z)$ is entire and z_0 lies inside C.

We have

$$f^{(2n)}(z_0) = \cos z^{(2n)}\Big|_{z=0} = (-1)^n,\ n = 0, 1, \ldots$$

Therefore

$$\oint_C \frac{\cos z}{z^{2n+1}}dz = (-1)^n\frac{2\pi i}{(2n)!}, \qquad n = 0, 1, \ldots$$

8 Her kan vi bruke Teorem 1 direkte med $n = 2$, $f(z) = z^3 + \sin z$ og $z_0 = i$, sidan f er analytisk i heile $\mathbb{C}$, og C omsluttar i.

$$\begin{aligned}\oint_C \frac{z^3+\sin z}{(z-i)^3}dz &= \oint_C \frac{f(z)}{(z-i)^3}dz \\ &= \frac{f''(i)2\pi i}{2!} \\ &= f''(i)\pi i\end{aligned}$$

Vi har $f''(z) = 6z - \sin z$, så $f''(i) = 6i - \sin i$. Svaret blir dermed

$$\begin{aligned}\oint_C \frac{z^3+\sin z}{(z-i)^3}dz &= (6i - \sin i)\pi i \\ &= -(6 + i\sin i)\pi\end{aligned}$$

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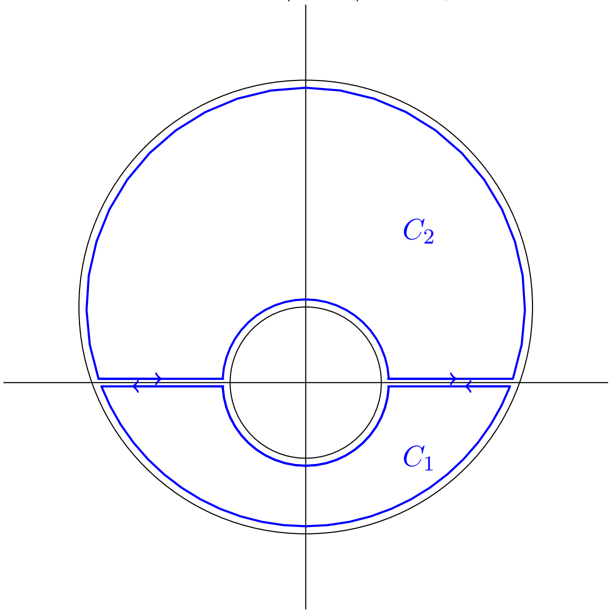
$$\oint_C \frac{e^{4z}}{z(z-2i)^2}dz \quad C : |z-i| = 3 \text{ mot uret}, |z| = 1 \quad \text{med uret}$$

$z = 0$ på innsiden av $|z-i| = 3$ og $|z| = 1$
$z = 2i$ på innsiden av $|z-i| = 3$ og på utsiden av $|z| = 1$



Vi deler opp mengden omsluttet av C i to enkeltsammenhengende mengder (se bildet).
$f(z) = \frac{e^{4z}}{z(z-2i)^2}$ er analytisk på innsiden av $C_1 + \epsilon$.
$g(z) = \frac{e^{4z}}{z}$ analytisk i $\mathbb{C}\backslash\{0\}$, $g'(z) = \frac{1}{z^2}(4e^{4z}z - e^{4z})$

$$\implies \oint_C \frac{e^{4z}}{z(z-2i)^2}dz = \oint_{C_2} \frac{e^{4z}}{z(z-2i)^2}dz = 2\pi i g'(2i) = 2\pi i \frac{1}{-4}(4e^{8i} \cdot 2i - e^{8i})$$

$$= -\frac{\pi i}{2}(8ie^{8i} - e^{8i}) = -\frac{\pi i}{2}e^{8i}(8i-1)$$