¡

Fra Kreyszig (10th), avsnitt 15.1

17 Ettersom $(-i)^{2n} = (-1)^n$ og $(-i)^{2n-1} = i(-1)^n$, vil annenhvert ledd i rekken være reelt og imaginært. Realdelen og imaginærdelen har skiftende fortegn og vil dermed konvergere ved alternerende rekketesten. Dvs. Rekken konvergerer (betinget) ved teorem 2.

$$\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln 2n} + i \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(2n-1)}$$

18 Vi skal undersøke om rekken

$$\sum_{n=1}^{\infty} n^2 \left(\frac{i}{4}\right)^n$$

er konvergent. Vi studerer forholdet ρ_n gitt ved

$$\rho_n = \left|\frac{z_{n+1}}{z_n}\right| = \frac{(n+1)^2 \left(\frac{1}{4}\right)^{n+1}}{n^2 \left(\frac{1}{4}\right)^n} = \frac{1}{4}\frac{(n+1)^2}{n^2}$$

når n går mot ∞. Denne grensen er $\frac{1}{4}$. Så rekken konvergerer.

24 Vi skal undersøke om rekken

$$\sum_{n=1}^{\infty} \left(\frac{(3i)^n n!}{n^n}\right)$$

er konvergent. Vi bruker forholdstesten (Theorem 8 på side 677 i læreboken)

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty} \left|\frac{(3i)^{n+1}(n+1)! n^n}{(n+1)^{n+1}}\right| = \lim_{n\to\infty} 3\left(\frac{n}{n+1}\right)^n = \frac{3}{e}.$$

Denne grensen er $\frac{3}{e} > 1$. Så rekken divergerer.

Fra Kreyszig (10th), avsnitt 15.2

10 The series

$$\sum_{n=0}^{\infty} \frac{(z-2i)^n}{n^n}$$

is a power series with center $z_0 = 2i$ and coefficients $a_n = \frac{1}{n^n}$. We use the Cauchy-Hadamard formula to determine the radius of convergence R.

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+1)^{n+1}}{n^n} = (n+1)\frac{(n+1)^n}{n^n} = (n+1)\underbrace{\left(1+\frac{1}{n}\right)^n}_{\xrightarrow[n\to\infty]{} e} \xrightarrow{n\to\infty} \infty.$$

Therefore $R = \infty$.

16 Senteret til rekken er 0.
Vi finner konvergensradiusen ved å bruke Cauchy-Hadamards formel.

$$R = \lim_{n=\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n=\infty} \frac{(3n)!2^{n+1}((n+1)!)^3}{2^n(n!)^3(3n+3)!}$$
$$= \lim_{n=\infty} \frac{2(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \frac{2}{27}$$

Fra Kreyszig (10th), avsnitt 15.3

5

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z-2i)^n$$

a) Cauchy-Hadamard.

$$R = \lim_{n\to\infty} \frac{n(n-1)}{4^n} \frac{4^{n+1}}{(n+1)n} = \lim_{n\to\infty} \frac{4(n-1)}{n+1} = 4$$

b) Alternativt.

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z-2i)^n = (z-2i)^2 \sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z-2i)^{n-2}$$

Denne summen svarer til to ganger leddvis derivasjon av serien

$$\sum_{n=0}^{\infty} \frac{(z-2i)^n}{4^n},$$

som konvergerer dersom

$$|\frac{(z-2i)}{4}| < 1 \iff |(z-2i)| < 4.$$

Dette gir igjen $R = 4$ siden den deriverte har samme konvergensradius.

8

$$\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n$$

a) Cauchy-Hadamard.

$$R = \lim_{n\to\infty} \frac{3^n}{n(n+1)} \frac{(n+1)(n+2)}{3^{n+1}} = \lim_{n\to\infty} \frac{n+2}{n} \frac{1}{3} = \frac{1}{3}$$

b) Alternativt.

$$\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n = \frac{3}{z} \sum_{n=1}^{\infty} \frac{3^{n-1}}{n(n+1)} z^{n+1} = \frac{3}{z} \sum_{n=0}^{\infty} \frac{3^n}{(n+1)(n+2)} z^{n+2}$$

To ganger leddvis derivasjon av serien

$$\sum_{n=0}^{\infty} \frac{3^n}{(n+1)(n+2)} z^{n+2},$$

gir serien

$$\sum_{n=0}^{\infty} 3^n z^n,$$

som konvergerer når $|3z| < 1 \iff |z| < \frac{1}{3}$. Dermed konvergerer

$$\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n,$$

når $|z| < \frac{1}{3}$. Altså igjen får vi at $R = \frac{1}{3}$.

Fra Kreyszig (10th), avsnitt 15.4

4 We write

$$\frac{z+2}{1-z^2} = \frac{(z+1)+1}{(1-z)(1+z)} = \frac{1}{1-z} + \frac{1}{1-z^2}.$$

The Maclaurin expansion of the first fraction in the sum is a geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

To find the Maclaurin expansion of $1/(1-z^2)$, we substitute z^2 for z in the geometric series and we get

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} (z^2)^n = \sum_{n=0}^{\infty} z^{2n} = 1 + z^2 + z^4 + \cdots$$

We obtain

$$\begin{aligned} \frac{z+2}{1-z^2} = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} z^{2n} &= 1 + z + z^2 + z^3 + \cdots + 1 + z^2 + z^4 + \cdots \\ &= 2 + z + 2z^2 + z^3 + 2z^4 + \cdots \end{aligned}$$

that is

$$\frac{z+2}{1-z^2} = \sum_{n=0}^{\infty} a_n z^n \qquad \text{with} \quad a_n = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 2 & \text{for } n \text{ even.} \end{cases}$$

Both the series $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=0}^{\infty} z^{2n}$ converge for $|z| < 1$, hence the whole series coverge for $|z| < 1$. This is consistent with the fact that the function $(z+2)/(1-z^2)$ is singular at $z = 1$ and $z = -1$, that have distance 1 from the center $z_0 = 0$.

8 Bruker rekken til $\cos z$ som konvergerer for alle z:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$
\begin{aligned}
\implies \qquad \sin^2 z &= \frac{1}{2} - \frac{1}{2}\cos 2z \\
&= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2z)^{2n}}{(2n)!} \\
&= \frac{1}{2} - \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n-1}z^{2n}}{(2n)!}
\end{aligned}
$$

La $w = 2z$. Rekka til $\cos w = \cos 2z$ konvergerer for alle w, så rekka ovanfor konvergerer for alle z.

23 La

$$f(z) = \frac{1}{(z-i)^2}, \qquad g(z) = -\frac{1}{z-i}.$$

Da er $g' = f$. Videre er

$$
\begin{aligned}
g(z) &= \frac{1}{i-z} \\
&= \frac{1}{i+i-(z+i)} \\
&= \frac{1}{2i\left(1-\frac{z+i}{2i}\right)} \\
&= \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{z+i}{2i}\right)^n \\
&= \frac{1}{2i}\sum_{n=0}^{\infty}\frac{1}{(2i)^n}(z+i)^n
\end{aligned}
$$

for $\left|\frac{z+i}{2i}\right| < 1$. Dvs. for $|z+i| < 2$ og dermed er $R = 2$.

Vi får nå at

$$
\begin{aligned}
\frac{1}{(z-i)^2} &= g'(z) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{2i}\sum_{n=0}^{\infty}\frac{1}{(2i)^n}(z+i)^n\right) \\
&= \frac{1}{2i}\sum_{n=1}^{\infty}\frac{n}{(2i)^n}(z+i)^{n-1} \\
&= \frac{1}{2i}\sum_{n=0}^{\infty}\frac{n+1}{(2i)^{n+1}}(z+i)^n \\
&= -\frac{1}{4}\sum_{n=0}^{\infty}\frac{n+1}{(2i)^n}(z+i)^n
\end{aligned}
$$

med konvergensradius $R = 2$.