Fra Kreyszig (10th), avsnitt 16.1

2

$$\text{Vet at } e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ er konvergent for alle komplekse tall } z$$

$$\implies \quad \frac{e^{-\frac{1}{z^2}}}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{\left(-1/z^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{1}{z^{2n+2}}$$

som konvergerer for alle $|z| > 0$. Dvs. $R = \infty$.

6 Let us find first the Laurent series of $1/z^2$ with center $z_0 = i$ and valid for $|z - z_0| < R$, with R to be determined. We observe that

$$\frac{1}{z^2} = \left(-\frac{1}{z}\right)'$$

and we also recall that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

Therefore we can write

$$-\frac{1}{z} = \frac{1}{-i-(z-i)} = \frac{1}{-i}\frac{1}{1-\left(\frac{z-i}{-i}\right)} = \frac{1}{-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{-i}\right)^n = \sum_{n=0}^{\infty}\frac{1}{(-i)^{n+1}}(z-i)^n$$

This series converges for $\left|\frac{z-i}{-i}\right| < 1$, that is, $|z-i| < 1$. Since the derived series has the same radius of convergence as the original series, we have

$$\left(-\frac{1}{z}\right)' = \frac{1}{z^2} = \sum_{n=1}^{\infty}\frac{n(z-i)^{n-1}}{(-i)^{n+1}} = \sum_{n=0}^{\infty}\frac{(n+1)(z-i)^n}{(-i)^{n+2}}, \qquad |z-i| < 1.$$

Notice that $1/z^2$ is analytic for $|z-i| < 1$ and in this region the Laurent series reduces to a Taylor series.

Finally, for the original function $1/[z^2(z-i)]$, we obtain

$$\frac{1}{z^2(z-i)} = \frac{1}{z-i}\sum_{n=0}^{\infty}\frac{(n+1)(z-i)^n}{(-i)^{n+2}} = \sum_{n=0}^{\infty}\frac{(n+1)(z-i)^{n-1}}{(-i)^{n+2}} = \sum_{n=-1}^{\infty}\frac{(n+2)(z-i)^n}{(-i)^{n+3}}$$
$$= -\frac{1}{z-i} - \sum_{n=0}^{\infty} i^{n+1}(n+2)(z-i)^n,$$

and the series is convergent for $0 < |z-i| < 1$.

Fra Kreyszig (10th), avsnitt 16.2

7

$$
\begin{aligned}
f(z) &= \frac{1}{(z+2i)^2} - \frac{z}{z-i} + \frac{z+1}{(z-i)^2} \\
&= \frac{1}{(z+2i)^2} + \frac{-z(z-i)+z+1}{(z-i)^2} \\
&= \frac{(z-i)^2 + (z+2i)^2(-z^2+zi+z+1)}{(z+2i)^2(z-i)^2}
\end{aligned}
$$

Uttrykket i nevneren er lik null for $z = -2i$ og $z = i$. For $z = -2i$ er uttrykket i telleren lik -9, mens for $z = i$ er uttrykket lik $-9-9i$, altså $\neq 0$ i begge tilfellene. Dermed har funksjonen $f(z)$ har en singularitet av andre orden i $z = -2i$ og en singularitet av andre orden i $z = i$.

9 Funksjonen

$$
f(z) = \frac{\sin z}{z-\pi}
$$

har en hevbar(removable) singularitet i $z = \pi$, fordi denne singulariteten kan fjernes ved å sette $f(\pi) = -1$, siden

$$
\begin{aligned}
\lim_{z\to\pi} \frac{\sin z}{z-\pi} &= \lim_{z\to\pi} \frac{\cos z}{1} \\
&= -1
\end{aligned}
$$

Fra Kreyszig (10th), avsnitt 16.3

4

$$
\begin{aligned}
f(z) &= \mathrm{e}^{1/(1-z)} \\
&= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{1-z}\right)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(z-1)^{-n} \\
&= 1 - \frac{1}{z-1} + \frac{1}{2(z-1)^2} - \frac{1}{6(z-1)^3} + \ldots
\end{aligned}
$$

Funksjonen $f(z)$ har en essensiell singularitet i $z = 1$. Residuen er koeffisienten foran $(z-1)^{-1}$ i laurentrekka:

$$
\operatorname*{Res}_{z=1} f(z) = -1
$$

6

$$
\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &= \oint_C \frac{z-23}{z^2-4z-5}\,\mathrm{d}z, \qquad C: \ |z-2-i| = 3.2 \\
&= \oint_C \frac{z-23}{(z-5)(z+1)}\,\mathrm{d}z
\end{aligned}
$$

Integranden har singulariteter i $z = 5$ og $z = -1$ av første orden, som begge ligger innenfor kurven C. Delbrøksoppspalter:

$$\frac{z-23}{(z-5)(z+1)} = \frac{-3}{z-5} + \frac{4}{z+1}$$

$$\begin{aligned}\oint_C \left(\frac{-3}{z-5} + \frac{4}{z+1}\right) \mathrm{d}z &= 2\pi i \left(\operatorname*{Res}_{z=5} f(z) + \operatorname*{Res}_{z=-1} f(z)\right) \\ &= 2\pi i \left(\lim_{z\to 5} \frac{z-23}{z+1} + \lim_{z\to -1} \frac{z-23}{z-5}\right) \\ &= 2\pi i(-3+4) \\ &= 2\pi i\end{aligned}$$

Fra Kreyszig (10th), avsnitt 16.4

9 Ser på

$$\begin{aligned}f(z) &= \frac{1}{z^4-1} \\ &= \frac{1}{(z^2+1)(z^2-1)} \\ &= \frac{1}{(z+i)(z-i)(z+1)(z-1)}\end{aligned}$$

Funksjonen har førsteordens pol i $z = \pm 1$ og $z = \pm i$. Av disse ligger $z = i$ i det øvre planet, mens $z = \pm 1$ ligger langs x-aksen. Vi får dermed

$$\begin{aligned}\text{pr. v.} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4-1} &= 2\pi i \operatorname*{Res}_{z=i} f(z) + \pi i \left(\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=1} f(z)\right) \\ &= 2\pi i \frac{1}{(i+i)(i+1)(i-1)} + \pi i \left(\frac{1}{(-1+i)(-i-1)(-1-1)} + \frac{1}{(1+i)(1-i)(1+1)}\right) \\ &= 2\pi i \left(\frac{1}{-4i}\right) + \pi i \left(\frac{-1}{4} + \frac{1}{4}\right) \\ &= -\frac{\pi}{2}\end{aligned}$$