From Kreyszig (10th), chapter 6.2

4 We take the laplace transform on both sides of the equation

$$y'' + 9y = 10e^{-t},$$

and obtain

$$\mathcal{L}(y'' + 9y) = \mathcal{L}(10e^{-t}).$$

By linearity of the Laplace transform and using that $y(0) =' (0) = 0$ we see that

$$\mathcal{L}(y'') = s^2Y - sy(0) - y'(0) = s^2Y,$$

and

$$\mathcal{L}(e^{-t}) = \frac{1}{s+1}.$$

Thus we need to solve the equation

$$s^2Y + 9Y = 10\frac{1}{s+1},$$

which is equivalent to

$$Y = 10\frac{1}{s+1} \cdot \frac{1}{s^2+9}. \tag{1}$$

To solve (1) we use partial fraction decomposition:

$$\frac{1}{s+1} \cdot \frac{1}{s^2+9} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9}$$
$$\implies 1 = (s^2+9)A + (Bs+C)(s+1)$$

Which leads to the following system of equations

$$\begin{aligned} 9A + C &= 1 \\ B + C &= 0 \\ A + B &= 0. \end{aligned}$$

We see that the solution is

$$A = -B = C = \frac{1}{10}.$$

Thus we may rewrite (1) as follows:

$$Y = \frac{1}{s+1} + \frac{1-s}{s^2+9}.$$

Finally, taking the inverse transform on both sides of the equality wee see that

$$\begin{aligned} y = \mathcal{L}^{-1}(y) &= \mathcal{L}^{-1}(\frac{1}{s+1}) + \frac{1}{3}\mathcal{L}^{-1}(\frac{3}{s^2+9}) - \mathcal{L}^{-1}(\frac{s}{s^2+9}) \\ &= e^{-t} + \frac{1}{3}\sin 3t - \cos 3t. \end{aligned}$$

From Kreyszig (10th), chapter 6.4

4 Laplace transforming

$$y'' + 16y = 4\delta(t - 3\pi)\,, \quad y(0) = 2\,,\ y'(0) = 0\,,$$

we get

$$\begin{aligned} & s^2Y - sy(0) - y'(0) + 16Y = 4e^{-3\pi s} \\ \Rightarrow\ & (s^2 + 16)Y = 2s + 4e^{-3\pi s}\,. \end{aligned}$$

Hence,

$$Y(s) = 2\frac{s}{s^2 + 4^2} + e^{-3\pi s}\frac{4}{s^2 + 4^2}\,.$$

Taking the inverse Laplace transform, using $t-$shifting, we obtain

$$\begin{aligned} y(t) = \mathcal{L}^{-1}(Y) &= 2\cos(4t) + \sin(4(t - 3\pi))u(t - 3\pi) \\ &= 2\cos(4t) + \sin(4t)u(t - 3\pi). \end{aligned}$$

10 Let us Laplace transform

$$y'' + 5y' + 6y = \delta\left(t - \frac{\pi}{2}\right) + u(t - \pi)\cos t, \quad y(0) = 0 = y'(0)\,.$$

Using $t-$shifting, we have that $\mathcal{L}\{u(t - \pi)\cos t\} = -e^{-\pi s}\frac{s}{s^2+1}$, since $\cos t = -\cos(t - \pi)$. Therefore

$$\begin{aligned} & s^2Y - sy(0) - y'(0) + 5sY - y(0) + 6Y = e^{-(\pi s)/2} - e^{-\pi s}\frac{s}{s^2 + 1} \\ \Rightarrow\ & \left(s^2 + 5s + 6\right)Y = e^{-(\pi s)/2} - e^{-\pi s}\frac{s}{s^2 + 1}\,. \end{aligned}$$

Hence,

$$Y(s) = e^{-(\pi s)/2}\frac{1}{s^2 + 5s + 6} - e^{-\pi s}\frac{s}{(s^2 + 1)\,(s^2 + 5s + 6)}.$$

Partial fraction decomposition.

$$\begin{aligned} & \frac{1}{s^2 + 5s + 6} = \frac{1}{(s + 2)(s + 3)} = \frac{A}{s + 2} + \frac{B}{s + 3} \\ \iff\ & 1 = A(s + 3) + B(s + 2) = (A + B)s + 3A + 2B. \end{aligned}$$

This yealds the following linear system for A and B:

$$\begin{cases} A + B = 0 \\ 3A + 2B = 1 \end{cases},$$

with solution $A = 1$ and $B = -1$.

Now

$$\begin{aligned} & \frac{s}{(s^2 + 1)\,(s + 2)(s + 3)} = \frac{As + B}{s^2 + 1} + \frac{C}{s + 2} + \frac{D}{s + 3} \\ \iff\ & As(s + 2)(s + 3) + B(s + 2)(s + 3) \\ & \quad + C\left(s^2 + 1\right)(s + 3) + D\left(s^2 + 1\right)(s + 2) \\ & = (A + C + D)s^3 + (5A + B + 3C + 2D)s^2 \\ & \quad + (6A + 5B + C + D)s + 6B + 3C + 2D\,. \end{aligned}$$

We obtain the following system of linear equations

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 5 & 1 & 3 & 2 \\ 6 & 5 & 1 & 1 \\ 0 & 6 & 3 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

with solution $A = B = 1/10, C = -2/5$ and $D = 3/10$.
Therefore, we have

$$\begin{aligned} Y(s) &= e^{-1/2\pi s} \frac{1}{(s+2)(s+3)} - e^{-\pi s} \frac{s}{(s^2+1)(s+2)(s+3)} \\ &= e^{-1/2\pi s} \left(\frac{1}{s+2} - \frac{1}{s+3} \right) - \frac{1}{10} e^{-\pi s} \left(\frac{s}{s^2+1} + \frac{1}{s^2+1} - 4\frac{1}{s+2} + 3\frac{1}{s+3} \right). \end{aligned}$$

Inverse Laplace transform, using $t-$shifting:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}(Y) &= \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)} \right) u(t - \pi/2) \\ &\quad - \frac{1}{10} \left(\cos(t-\pi) + \sin(t-\pi) - 4e^{-2(t-\pi)} + 3e^{-3(t-\pi)} \right) u(t-\pi) \\ &= \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)} \right) u(t - \pi/2) \\ &\quad + \frac{1}{10} \left(\cos t + \sin t + 4e^{-2(t-\pi)} - 3e^{-3(t-\pi)} \right) u(t-\pi). \end{aligned}$$

From Kreyszig (10th), chapter 6.5

12 We can rewrite the equation as

$$y(t) + (y * \cosh)(t) = t + e^t.$$

Laplace transforming and applying the convolution theorem, we obtain

$$\begin{aligned} Y + Y\frac{s}{s^2-1} &= \frac{1}{s^2} + \frac{1}{s-1}, \\ \Rightarrow \quad Y(s) &= \frac{1}{s} + \frac{1}{s^2}, \end{aligned}$$

with $Y(s) = \mathcal{L}(y(t))$. Inverse Laplace transform:

$$y(t) = 1 + t.$$

19 The exercise asks us to find $f(t)$ when

$$\mathcal{L}(f) = \frac{2\pi s}{(s^2+\pi^2)^2}.$$

We observe that

$$\mathcal{L}(f) = 2\frac{\pi}{s^2+\pi^2}\frac{s}{s^2+\pi^2} = 2\mathcal{L}(\sin \pi t)\mathcal{L}(\cos \pi t).$$

Hence it follows that

$$\begin{aligned} f(t) &= 2 \sin \pi t * \cos \pi t \\ &= 2 \int_0^t \sin(\pi\tau) \cos(\pi(t-\tau)) d\tau. \end{aligned}$$

Using the trigonometric formulas for sums and the half-angle formulas we obtain

$$\begin{aligned}2\sin\pi\tau\cos(\pi t-\pi\tau) &= 2\sin\pi\tau(\cos\pi t\cos\pi\tau+2\sin\pi t\sin\pi\tau)\\ &= 2\cos\pi t\sin\pi\tau\cos\pi\tau+2\sin\pi t\sin^2\pi\tau\\ &= \cos\pi t\sin 2\pi\tau+\sin\pi t(1-\cos 2\pi\tau)\\ &= \sin\pi t+\sin 2\pi\tau\cos\pi t-\cos 2\pi\tau\sin\pi t\\ &= \sin\pi t+\sin(2\pi\tau-\pi t).\end{aligned}$$

It thus follows that

$$\begin{aligned}f(t) &= \int_0^t \sin\pi t+\sin(2\pi\tau-\pi t)d\tau\\ &= \sin\pi t\Big|_0^t\tau-\frac{1}{2\pi}\Big|_0^t\cos(2\pi\tau-\pi t)\\ &= t\sin\pi t-\frac{1}{2\pi}(\cos\pi t-\cos(-\pi t))\\ &= t\sin\pi t.\end{aligned}$$

Note that an alternative way to solve this problem is to observe that

$$\frac{2\pi s}{(s^2+\pi^2)^2}=\left(-\frac{\pi}{s^2+\pi^2}\right)',<$$

and then use the formula for the derivative of the Laplace transform.

22 We have

$$\begin{aligned}F(s)=\mathcal{L}\{f\} &= e^{-as}\mathcal{L}\{1\}\mathcal{L}\left\{e^{2t}\right\}\\ &= \mathcal{L}\{u(t-a)\}\mathcal{L}\left\{e^{2t}\right\}\\ &= \mathcal{L}\left\{u(t-a)*e^{2t}\right\}.\end{aligned}$$

Thus it follows that

$$\begin{aligned}f(t) &= u(t-a)*e^{2t}\\ &= \int_0^t u(\tau-a)e^{2(t-\tau)}\mathrm{d}\tau\\ &= u(t-a)\int_a^t e^{2(t-\tau)}\mathrm{d}\tau\\ &= -\frac{1}{2}u(t-a)e^{2t}e^{-2\tau}\Big|_a^t\\ &= -\frac{1}{2}u(t-a)e^{2t}\left(e^{-2t}-e^{-2a}\right)\\ &= \frac{1}{2}u(t-a)\left(e^{2(t-a)}-1\right).\end{aligned}$$

From Kreyszig (10th), chapter 6.6

6 We use the formula

$$\mathcal{L}\{tf(t)\}=-F'(s)$$

twice:

$$g(t) = \sin(3t) \implies G(s) = \mathcal{L}[g](s) = \frac{3}{s^2+9}$$

$$\begin{aligned}
&\implies \mathcal{L}[t\sin(3t)](s) = -G'(s) = \frac{3}{(s^2+9)^2}2s = \frac{6s}{(s^2+9)^2} \\
&\implies \mathcal{L}[t^2\sin(3t)](s) = -\frac{6}{(s^2+9)^4}((s^2+9)^2 - s2(s^2+9)2s) \\
&\quad = -\frac{6}{(s^2+9)^3}(s^2+9-4s^2) = 18\frac{s^2-3}{(s^2+9)^3} \\
&\implies \mathcal{L}[f](s) = 18\frac{s^2-3}{(s^2+9)^3}
\end{aligned}$$

7 We observe that

$$\begin{aligned}
\mathcal{L}\{t\sinh 2t\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{2}{s^2-4} \\
&= \frac{4s}{\left(s^2-4\right)^2}.
\end{aligned}$$

Then using the formula

$$\mathcal{L}\{tf(t)\} = -F'(s)$$

it follows that

$$\begin{aligned}
\mathcal{L}\{f\} &= \mathcal{L}\{t\cdot t\sinh 2t\} \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{4s}{\left(s^2-4\right)^2} \\
&= -\frac{4\left(s^2-4\right)^2 - 4s\cdot 2\left(s^2-4\right)2s}{\left(s^2-4\right)^4} \\
&= 4\frac{4+3s^2}{\left(s^2-4\right)^3}.
\end{aligned}$$