From Kreyszig (10th), section 6.7

4 Writing $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$, $G_1 = \mathcal{L}(\cos 4t)$ and $G_2 = \mathcal{L}(\sin 4t)$ we obtain

$$\begin{aligned} sY_1 - y_1(0) &= 4Y_2 - 8G_1 \\ sY_2 - y_2(0) &= -3Y_1 - 9G_2, \end{aligned}$$

with $y_1(0) = 0$ and $y_2(0) = 3$. By collecting Y_1 and Y_2-terms we have

$$\begin{aligned} sY_1 - 4Y_2 &= -8G_1 \\ 3Y_1 + sY_2 &= 3 - 9G_2. \end{aligned}$$

Solving algebraically for Y_1 and Y_2 we get

$$\begin{aligned} Y_1 &= \frac{1}{s^2+12}(12 - 8sG_1 - 36G_2) \\ Y_2 &= \frac{1}{s^2+12}(3s + 24G_1 - 9sG_2). \end{aligned}$$

Substituting $G_1 = \frac{s}{s^2+16}$ and $G_2 = \frac{4}{s^2+16}$ yealds

$$\begin{aligned} Y_1 &= \frac{1}{s^2+12}\left(12 - \frac{8s^2}{s^2+16} - \frac{144}{s^2+16}\right) = \frac{1}{s^2+12}\left(\frac{12s^2+192-8s^2-144}{s^2+16}\right) \\ &= \frac{1}{s^2+12}\left[\frac{4(s^2+12)}{s^2+16}\right] = \frac{4}{s^2+16}. \end{aligned}$$

Inverse transform:

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \sin(4t).$$

We can proceed in the same way to find $y_2(t)$. We have

$$\begin{aligned} Y_2 &= \frac{1}{s^2+12}\left(3s + \frac{24s}{s^2+16} - \frac{36s}{s^2+16}\right) = \frac{1}{s^2+12}\left(3s - \frac{12s}{s^2+16}\right) \\ &= \frac{1}{s^2+12}\frac{3s(s^2+12)}{s^2+16} = \frac{3s}{s^2+16}, \end{aligned}$$

hence

$$y_2(t) = \mathcal{L}^{-1}(Y_2) = 3\cos(4t).$$

Alternatively, since we had found y_1 already, we could have solved

$$y_2' = -3y_1 - 9\sin 4t = -12\sin 4t\,, \quad y_2(0) = 3,$$

from which

$$y_2(t) = y_2(0) - 12\int_0^t \sin(4\tau)\,d\tau = 3 + 3\cos(4\tau)\Big|_0^t = 3\cos(4t).$$

6 We are solving the following IVP.

$$y_1' = 5y_1 + y_2 \qquad y_1(0) = 1$$
$$y_2' = y_1 + 5y_2 \qquad y_2(0) = -3$$

We let $Y_1 = \mathcal{L}[y_1]$ and $Y_2 = \mathcal{L}[y_2]$. This then leads to

$$sY_1 - y_1(0) = 5Y_1 + Y_2$$
$$sY_2 - y_2(0) = Y_1 + 5Y_2$$

$$\implies (5-s)Y_1 + Y_2 = -1$$
$$Y_1 + (5-s)Y_2 = 3$$

$$\implies (5-s)Y_1 + Y_2 = -1$$
$$(5-s)Y_1 + (5-s)^2Y_2 = 3(5-s)$$

$$\implies ((s-5)^2 - 1)Y_2 = 3(5-s) + 1$$

$$\implies Y_2(s) = 3\frac{5-s}{(s-5)^2-1} + \frac{1}{(s-5)^2-1} = -3\mathcal{L}[e^{5t}\cosh t](s) + \mathcal{L}[e^{5t}\sinh t)](s)$$
$$Y_1(s) = 3 - 3\frac{(5-s)^2}{(s-5)^2-1} - \frac{(5-s)}{(s-5)^2-1} = -3\frac{1}{(s-5)^2-1} + \frac{(s-5)}{(s-5)^2-1}$$
$$= \mathcal{L}[-3e^{5t}\sinh t + e^{5t}\cosh t](s)$$

$$\implies y_1(t) = -3e^{5t}\sinh t + e^{5t}\cosh t$$
$$y_2(t) = -3e^{5t}\cosh t + e^{5t}\sinh t$$

From Kreyszig (10th), section 11.1

2 $\frac{2\pi}{n}$, $\frac{2\pi}{n}$, k, k, $\frac{k}{n}$, $\frac{k}{n}$

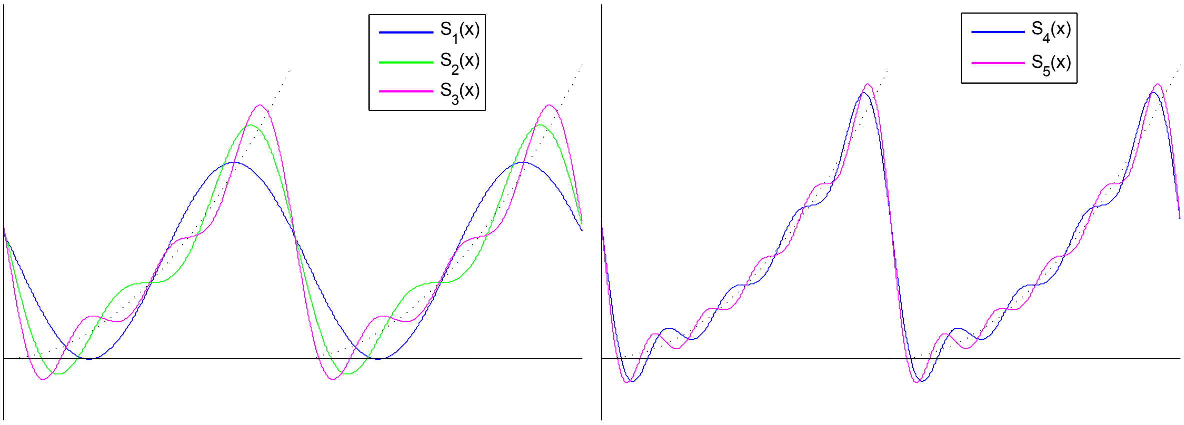
15 We start by transforming the function by $-\pi$ so that the domain the function is defined for is symmetric around 0. We then calculate the coefficients by using partial integration, and using symmetry around 0 to see that some integrals are 0.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+\pi)^2\mathrm{d}x = \frac{1}{2\pi}\Big[\frac{x^3}{3} + \pi x^2 + \pi^2 x\Big]_{-\pi}^{\pi} = \frac{4}{3}\pi^2$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)^2\cos nx\ \mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 + 2\pi x + \pi^2)\cos nx\ \mathrm{d}x$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\ \mathrm{d}x = \Big[\frac{n^2x^2\sin nx - 2\sin nx + 2nx\cos nx}{\pi n^3}\Big]_{-\pi}^{\pi} = 4\frac{(-1)^n}{n^2},\ n \geq 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)^2\sin nx\ \mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 + 2\pi x + \pi^2)\sin nx\ \mathrm{d}x$$
$$= 2\int_{-\pi}^{\pi}x\sin nx\ \mathrm{d}x = 4\Big[\frac{\sin nx - nx\cos nx}{n^2}\Big]_0^{\pi} = 4\pi\frac{(-1)^{n+1}}{n},\ n \geq 1$$

We thus obtain the following Fourier series for $f(x)$:

$$f(x) = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2} \cdot \cos n(x-\pi) + 4\pi\frac{(-1)^{n+1}}{n} \cdot \sin n(x-\pi)$$



Alternatively we can express the Fourier series as follows:

$$\begin{aligned} f(x) &= \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(nx) - \frac{4\pi}{n}\sin(nx) \\ &= \frac{4}{3}\pi^2 + 4\left(\cos x + \frac{1}{4}\cos(2x) + \frac{1}{9}\cos(3x) + ...\right) - 4\pi\left(\sin x + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + ...\right) \end{aligned}$$

We see from the graphs that the more terms in the sum we include, the closer we come to the original function.

17 We start by finding $f(x)$:

$$f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$$

Then we calculate the Fourier coefficients using that f is an even function.

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{2\pi}\cdot\frac{1}{2}\cdot 2\pi\cdot\pi = \frac{\pi}{2} \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{d}x \\ &= \frac{2}{\pi}\int_{-\pi}^{0}(\pi + x)\cos nx\ \mathrm{d}x = \left[\frac{n\pi\sin nx + nx\sin nx + \cos nx}{n^2}\right]_{-\pi}^{0} \\ &= \frac{2(1-\cos\pi n)}{\pi n^2} = \frac{2(1-(-1)^n)}{\pi n^2},\ n \ge 1 \end{aligned}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{d}x \overset{\sin\text{ odde}}{=} 0,\ n \ge 1$$

Thus the series is as follows:

$$f(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{4}{\pi(2n+1)^2}\cos(2n+1)x.$$

21 We start by writing down the function f:

$$f(x) = \begin{cases} -\pi - x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

We calculate the coefficients using that f is an even function

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0 \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x-\pi)\cos nx dx + \int_0^{\pi}(-x+\pi)\cos nx dx\right) \end{aligned}$$

substituting $y = -x$ in the first integral we obtain

$$a_n = 0.$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x-\pi)\sin nx dx + \int_0^{\pi}(-x+\pi)\sin nx dx\right) \end{aligned}$$

substituting $y = -x$ in the first integral we obtain

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi}(-x+\pi)\sin nx dx \\ &= \frac{2}{\pi}\left(\Big|_0^{\pi}\frac{x\cos nx}{n} + \frac{1}{n}\int_0^{\pi}\cos nx dx + \pi\int_0^{\pi}\sin nx\right) \\ &= \frac{2}{\pi}\left(\frac{\pi\cos n\pi}{n} - \frac{1}{n^2}\Big|_0^{\pi}\sin nx - \frac{\pi}{n}\Big|_0^{\pi}\cos nx\right) \\ &= \frac{2}{\pi}\left(\frac{\pi}{n}\cos n\pi - 0 - \frac{\pi}{n}(\cos n\pi - 1)\right) \\ &= \frac{2}{n}. \end{aligned}$$

We thus obtain the following Fourier series

$$a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

From Kreyszig (10th), section 11.2

1 $f(x) = e^x$ is neither even nor odd er this:

$$\begin{aligned} f(-1) &= e^{-1} \\ &= \frac{1}{e} \\ &\neq \begin{cases} -e & = -f(1) \\ e & = f(1). \end{cases} \end{aligned}$$

b) $f(x) = e^{-|x|}$ is even since

$$f(-x) = e^{-|-x|} = e^{-|x|} = f(x).$$

c) $f(x) = x^3 \cos nx$ is odd since

$$\begin{aligned} f(-x) &= (-x)^3 \cos(-nx) \\ &= -x^3 \cos nx \\ &= -f(x) \end{aligned}$$

d) $f(x) = x^2 \tan \pi x$ is odd since $\tan x$ is odd and

$$f(-x) = (-x)^2 \tan(-\pi x) = -x^2 \tan \pi x = -f(x).$$

e) $f(x) = \sinh x - \cosh x$is neither odd nor even since

$$\begin{aligned} f(x) &= \sinh x - \cosh x \\ &= \frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2} \\ &= -e^{-x} \end{aligned}$$

9 The function is odd with period $P = 2L = 4$. Thus $a_n = 0$ for $n = 0, 1, 2, ...$ Further we calculate that

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^{2} f(x) \sin \frac{n\pi x}{2} dx = \int_0^2 f(x) \sin \frac{n\pi x}{2} dx \\ &= \int_0^2 \sin \frac{n\pi x}{2} dx = \frac{2}{n\pi} \left[-\cos \frac{n\pi x}{2} \right]_0^2 = \frac{2}{n\pi} (1 - (-1)^n) \end{aligned}$$

We have here uses that $f \cdot \sin$ is even and that $\cos n\pi = (-1)^n$). Hence it follows that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} = \frac{4}{\pi} \left(\sin \frac{\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} + \frac{1}{5} \sin \frac{5\pi x}{2} + ... \right)$$

17 The function is even and has period $P = 2L = 2$. Thus $b_n = 0$ for $n = 1, 2, 3, ...$
Since $f(x) = 1 - |x|$ for $x \in [-1, 1]$, it follows that

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(x) dx \overset{\text{even}}{=} \int_0^1 f(x) dx = \int_0^1 (1 - x) dx = \frac{1}{2}$$

$$\begin{aligned} a_n &= \frac{1}{1} \int_{-1}^{1} f(x) \cos \frac{n\pi x}{1} dx \\ &\overset{\text{even}}{=} 2 \int_0^1 f(x) \cos n\pi x dx \\ &= 2 \int_0^1 (1 - x) \cos n\pi x dx \\ &= 2 \left[(1 - x) \frac{1}{n\pi} \sin n\pi x \right]_0^1 - 2 \int_0^1 (-1) \frac{1}{n\pi} \sin n\pi x dx \\ &= 0 + \frac{2}{n\pi} \left[\frac{-1}{n\pi} \cos n\pi x \right]_0^1 \\ &= \frac{2}{(n\pi)^2} (1 - (-1)^n) \end{aligned}$$

Thus we obtain the following series:

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x \\
&= \frac{1}{2} + \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + ...\right)
\end{aligned}
$$

25 We start with the cosinus-part

$$
a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \pi - x\ \mathrm{d}x = \frac{1}{\pi}\Big[\pi x - \frac{1}{2}x^2\Big]_0^{\pi} = \underline{\frac{1}{2}\pi}
$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\ \mathrm{d}x \\
&= \frac{1}{\pi}\Big[\frac{\pi}{n}\sin nx - \frac{nx\sin nx + \cos nx}{n^2}\Big]_0^{\pi} = \underline{\frac{2(1+(-1)^{n+1})}{\pi n^2}},\ n \geq 1 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{d}x = \underline{0},\ n \geq 1 \\
\Rightarrow f(x) &= \underline{\frac{1}{2}\pi + \sum_{n=0}^{\infty}\frac{4}{\pi(2n+1)^2}\cos(2n+1)x}
\end{aligned}
$$

Then we calculate the sinus-part:

$$
a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x \overset{\text{f odde}}{=} \underline{0}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{d}x \overset{\text{f odde}}{=} \underline{0} \\
b_n &= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin nx\ \mathrm{d}x \\
&= \frac{2}{\pi}\Big[-\frac{\pi}{n}\cos nx + \frac{\sin nx + nx\cos nx}{n^2}\Big]_0^{\pi} = \underline{\frac{2}{n}},\ n \geq 1 \\
\Rightarrow f(x) &= \underline{\sum_{n=1}^{\infty}\frac{2}{n}\sin nx}
\end{aligned}
$$