Fra Kreyszig (10th), avsnitt 11.4

2 Vi vet at koeffisientene som minimerer $\int_{-\pi}^{\pi}(f(x) - F(x))^2 \, \mathrm{d}x$ er Fourier-koeffisientene til f. Ettersom f er odde, finner vi at $A_n = 0$ og

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin nx \, \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi x\sin nx \, \mathrm{d}x \\
&= \frac{2}{\pi}\left(- |_0^\pi \, x\frac{\cos nx}{n} + \frac{1}{n}\int_0^\pi \cos nx \, \mathrm{d}x\right) \\
&= \frac{2}{\pi}\left(-\pi\frac{(-1)^n}{n} + 0\right) \\
&= (-1)^{n+1}\frac{2}{n}
\end{aligned}$$

Dvs.

$$x = f(x) \approx F_N(x) = 2\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n}\sin nx$$

på $(-\pi, \pi)$. Feilen er gitt ved

$$\begin{aligned}
E_N &= \int_{-\pi}^{\pi}(f(x))^2 \, \mathrm{d}x - \pi\left(2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right) \\
&= 2\int_0^\pi x^2 \, \mathrm{d}x - \pi\sum_{n=1}^{N} b_n^2 \\
&= \frac{2}{3}\pi^3 - 4\pi\sum_{n=1}^{N}\frac{1}{n^2} = 4\pi\left(\frac{\pi^2}{6} - \sum_{n=1}^{N}\frac{1}{n^2}\right).
\end{aligned}$$

Dette gir

$$\begin{aligned}
E_1 &= 4\pi\left(\frac{\pi^2}{6} - 1\right) \\
&\approx 8.10,
\end{aligned}$$

$$\begin{aligned}
&\approx 8.10, \\
E_2 &= 4\pi\left(\frac{\pi^2}{6} - 1 - 1/4\right) \\
&\approx 4.96, \\
E_3 &= 4\pi\left(\frac{\pi^2}{6} - 1 - 1/4 - 1/9\right) \\
&\approx 3.57,
\end{aligned}$$

$$
\begin{aligned}
E_4 &= 4\pi\left(\frac{\pi^2}{6} - 1 - 1/4 - 1/9 - 1/16\right) \\
&\approx 2.78, \\
E_5 &= 4\pi\left(\frac{\pi^2}{6} - 1 - 1/4 - 1/9 - 1/16 - 1/25\right) \\
&\approx 2.28.
\end{aligned}
$$

3 Vi vet at koeffisientene som minimerer $\int_{-\pi}^{\pi}(f(x) - F(x))^2 \,\mathrm{d}x$ er Fourier-koeffisientene til f. Ettersom f er jevn, finner vi at $B_n = 0$,

$$
A_0 = \frac{1}{\pi}\int_0^\pi |x|\mathrm{d}x = \frac{1}{2\pi}\pi^2 = \frac{\pi}{2}
$$

og

$$
\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^\pi x\cos nx \,\mathrm{d}x \\
&= \frac{2}{\pi}\left(|_0^\pi x\frac{\sin nx}{n} - \frac{1}{n}\int_0^\pi \sin nx \,\mathrm{d}x\right) \\
&= \frac{2}{\pi}\left(0 + \frac{1}{n}\bigg|_0^\pi \frac{\cos nx}{n}\right) \\
&= \frac{2}{n^2\pi}\left((-1)^n - 1\right) \\
&= \begin{cases} -\frac{4}{n^2\pi}, & n \text{ odde} \\ 0, & n \text{ jevn.} \end{cases}
\end{aligned}
$$

Dvs.

$$
|x| = f(x) \approx F_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{(N+1)/2}\frac{\cos(2n-1)x}{(2n-1)^2}
$$

for N odde og $F_N = F_{N-1}$ for N jevn. Feilen er gitt ved

$$
\begin{aligned}
E_N &= \int_{-\pi}^\pi (f(x))^2 \,\mathrm{d}x - \pi\left(2a_0^2 + \sum_{n=1}^N \left(a_n^2 + b_n^2\right)\right) \\
&= 2\int_0^\pi x^2 \,\mathrm{d}x - \pi\left(\frac{\pi^2}{2} + \sum_{n=1}^{(N+1)/2} a_{2n-1}^2\right), \quad N \text{ odd} \\
&= \frac{\pi^3}{6} - \frac{16}{\pi}\sum_{n=1}^{(N+1)/2}\frac{1}{(2n-1)^4} \\
&= \frac{16}{\pi}\left(\frac{\pi^4}{96} - \sum_{n=1}^{(N+1)/2}\frac{1}{(2n-1)^4}\right)
\end{aligned}
$$

og $E_N = E_{N-1}$ for N jevn. Dette gir

$$\begin{aligned} E_1 &= \frac{16}{\pi}\left(\frac{\pi^4}{96} - 1\right) \\ &\approx 0.0748 \\ E_2 &= E_1 \\ E_3 &= \frac{16}{\pi}\left(\frac{\pi^4}{96} - 1 - 1/3^4\right) \\ &\approx 0.01187 \\ E_4 &= E_3 \\ E_5 &= \frac{16}{\pi}\left(\frac{\pi^4}{96} - 1 - 1/3^4 - 1/5^4\right) \\ &\approx 0.00373. \end{aligned}$$

13 Skal ved hjelp av oppgave 11.1.17 vise at

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ... = \frac{\pi^4}{96}$$

Oppgave 11.1.17 sier at Fourier-rekka til

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

er

$$F(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + ...\right)$$

Ser at koeffisientene må være

$$\begin{aligned} a_0 &= \frac{\pi}{2} \\ a_n &= \frac{2}{\pi n^2}(1 - (-1)^n) \\ b_n &= 0 \end{aligned}$$

Trikset her er å sette disse uttrykkene inn i Parsevals identitet:

$$\begin{aligned} 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \, \mathrm{d}x \\ \frac{\pi^2}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}(1 - (-1)^n)\right)^2 &= \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)^2 \, \mathrm{d}x \\ \frac{\pi^2}{2} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1 - (-1)^n}{n^4} &= \frac{2}{\pi}\left(\pi^3 - \pi^3 + \frac{\pi^3}{3}\right) \\ \sum_{n=1}^{\infty}\frac{1 - (-1)^n}{n^4} &= \frac{\pi^4}{48} \\ 2\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ...\right) &= \frac{\pi^4}{48} \end{aligned}$$

Som gir det endelige svaret:

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ... = \frac{\pi^4}{96}$$

9-utg9 Fra oppgave 11.4.2 er

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

der $b_n = (-1)^{n+1}\frac{2}{n}$. Vi ønsker å finne de komplekse koeffisientene c_n slik at

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Skriv Vi har at

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{inx} + c_{-n} e^{-inx}\right)$$

$$e^{inx} = \cos nx + i \sin nx$$

$$e^{-inx} = \cos nx - i \sin nx$$

så dermed må $c_0 = 0$ og

$$b_n \sin nx = (c_n + c_{-n}) \cos nx + i\,(c_n - c_{-n}) \sin nx.$$

Dvs. $c_n = -c_{-n}$ og

$$b_n = i\,(c_n - c_{-n}) = 2ic_n.$$

Dette gir $c_n = \frac{b_n}{2i} = -i\frac{b_n}{2} = i\frac{(-1)^n}{n}$ og Fourier-rekken

$$f(x) = i \sum_{n=-\infty, n\neq 0}^{\infty} \frac{(-1)^n}{n} e^{inx}.$$

11-utg9 Vi skal finne den komplekse Fourierrekken til $f(x) = x^2$ for $-\pi < x < \pi$. Vi bruker formelen

$$c_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Vi starter med $n = 0$:

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}.$$

Vi betrakter deretter $n \neq 0$:

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{inx} dx \\
&= \frac{1}{2\pi} \left[\frac{x^2}{-in} e^{-inx}\right]_{\pi}^{\pi} + \frac{1}{i\pi n} \int_{-\pi}^{\pi} x e^{-inx} dx \\
&= \frac{1}{2\pi} \left(\frac{\pi^2}{in} e^{in\pi} - \frac{\pi^2}{in} e^{-in\pi}\right) + \frac{1}{i\pi n} \left[\frac{x}{-in} e^{-inx}\right]_{-\pi}^{\pi} + \frac{1}{\pi n^2} \int_{-\pi}^{\pi} e^{-inx} dx \\
&= \frac{1}{\pi n} \left(\frac{\pi}{n} e^{-in\pi} + \frac{\pi}{n} e^{in\pi}\right) - \frac{1}{\pi n^2} \frac{1}{in} [e^{-inx}]_{-\pi}^{\pi} \\
&= 2\frac{(-1)^n}{n^2} - \frac{1}{\pi i n^3} (e^{-in\pi} - e^{in\pi}) \\
&= 2\frac{(-1)^n}{n^2}.
\end{aligned}$$

Her har vi brukt at $e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n$. Dermed har $f(x)$ den komplekse Fourierrekken

$$f(x) = \frac{\pi^2}{3} + \sum_{n \in Z \setminus 0} \frac{2(-1)^n}{n^2} e^{inx}.$$

Vi ser at $c_n = c_{-n}$ og vi har dermed at

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} e^{inx}.$$