Løsningsforslag - Øving 5

Fra Kreyszig (10th), avsnitt 11.7

1 Definér funksjonen f på $\mathbb{R}$ ved $f(x) = \pi e^{-x}$ for $x \geq 0$ og $f(x) = 0$ for $x < 0$. Da er f lik sitt Fourier-integral for alle x bortsett fra diskontinuiteten i $x = 0$. Dvs

$$f(x) = \int_0^\infty (A(w)\cos wx + B(w)\sin wx)\mathrm{d}w, \quad x \neq 0.$$

Nå er

$$\begin{aligned}
A(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt \ \mathrm{d}t \\
&= \int_0^\infty e^{-t}\cos wt \ \mathrm{d}t \\
&= |_{s=1} \int_0^\infty e^{-st}\cos wt \ \mathrm{d}t \\
&= \mathcal{L}\{\cos wt\}(1) \\
&= |_{s=1} \frac{s}{s^2+w^2} \\
&= \frac{1}{1+w^2}
\end{aligned}$$

og

$$\begin{aligned}
B(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt \ \mathrm{d}t \\
&= \int_0^\infty e^{-t}\sin wt \ \mathrm{d}t \\
&= |_{s=1} \int_0^\infty e^{-st}\sin wt \ \mathrm{d}t \\
&= \mathcal{L}\{\sin wt\}(1) \\
&= |_{s=1} \frac{w}{s^2+w^2} \\
&= \frac{w}{1+w^2}
\end{aligned}$$

I $x = 0$ vil verdien av integralet være gjennomsnittet av grenseverdiene fra høyre og venstre. Dvs.

$$\int_0^\infty A(w)\mathrm{d}w = \frac{\lim_{x\to 0-f} f(x) + \lim_{x\to 0+} f(x)}{2} = \frac{0+\pi}{2}.$$

Dermed er

$$\begin{aligned}\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}\,\mathrm{d}w &= \int_0^\infty (A(w)\cos wx + B(w)\sin wx)\mathrm{d}w \\ &= \begin{cases} f(x), & x \neq 0 \\ \pi/2, & x = 0 \end{cases} \\ &= \begin{cases} 0, & x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x}, & x > 0. \end{cases}\end{aligned}$$

11 La $g : \mathbb{R} \to \mathbb{R}$ vere gitt ved $g(x) = f(|x|)$. Sidan g tilfredsstiller krava i Teorem 1 og er like, er $g(x)$, og dermed $f(x)$, gitt ved (10) på side 515. Reknar ut $A(w)$:

$$\begin{aligned}A(w) &= \frac{2}{\pi}\int_0^\infty g(v)\cos wv dv \\ &= \frac{2}{\pi}\int_0^\pi \sin v \cos wv dv \\ &= \frac{2}{\pi}\left[\frac{w\sin v \sin wv + \cos v \cos wv}{w^2-1}\right]_0^\pi \\ &= \frac{2\cos w\pi + 2}{\pi(1-w^2)}.\end{aligned}$$

Vi får dermed

$$f(x) = \int_0^\infty \frac{2\cos w\pi + 2}{\pi(1-w^2)}\cos wx dw$$

19 Vi betrakter f som en del av en odde funksjon definert på hele $\mathbb{R}$. Dermed er f gitt ved (11) fra side 515 i boken, der

$$\begin{aligned}B(w) &= \frac{2}{\pi}\int_0^\infty f(v)\sin wv dv \\ &= \frac{2}{\pi}\int_0^1 e^v \sin wv dv \\ &= \frac{2}{\pi}\left([e^v \sin wv]_0^1 - \int_0^1 we^v \cos wv dv\right) \\ &= \frac{2}{\pi}\left(e\sin w - [e^v w\cos wv]_0^1 - \int_0^1 w^2 e^v \sin wv dv\right) \\ &= \frac{2}{\pi}\left(e\sin w + w - ew\cos w - w^2\int_0^1 e^v \sin wv dv\right) \\ \Rightarrow B(w) &= \frac{2}{\pi}\cdot\frac{w + e(\sin w - w\cos w)}{1+w^2}.\end{aligned}$$

Det gir

$$f(x) = \int_0^\infty \frac{2(w + e(\sin w - w\cos w))}{\pi(1+w^2)}\sin wx dw$$

Fra Kreyszig (10th), avsnitt 11.9

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$$f(x) = \begin{cases} e^x & \text{for } -a < x < a \\ 0 & \text{ellers} \end{cases}$$

Fouriertransformasjon:

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-iwx} dx \\ \implies \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^x e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{(1-iw)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{1-iw} e^{(1-iw)x} \Big|_{-a}^{a} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{1-iw} (e^{(1-iw)a} - e^{-(1-iw)a}) \end{aligned}$$

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$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{a} x e^{-iwx} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \left(- |_0^a \, x \frac{e^{-iwx}}{iw} + \frac{1}{iw} \int_0^a e^{-iwx} \, \mathrm{d}x \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{iw} a e^{-iwa} - \frac{1}{(iw)^2} \Big|_0^a e^{-iwx} \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{i}{w} a e^{-iwa} + \frac{1}{w^2} \left(e^{-iwa} - 1 \right) \right) \\ &= \frac{(iaw+1)e^{-iwa} - 1}{\sqrt{2\pi} w^2}. \end{aligned}$$

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$$f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{ellers} \end{cases}$$

Bruker definisjonen på Fourier-transform:

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} x e^{-x} e^{-i\omega x} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} x e^{(-1-i\omega)x} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{1}{-1-i\omega} x e^{(-1-i\omega)x} \right]_{-1}^{0} - \frac{1}{-1-i\omega} \int_{-1}^{0} e^{(-1-i\omega)x} \mathrm{d}x \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{1+i\omega} e^{(1+i\omega)} - \frac{1}{(1+i\omega)^2} \left(1 - e^{(1+i\omega)} \right) \right) \\
&= \underline{-\frac{1}{\sqrt{2\pi}} \frac{1}{1+iw} e^{1+iw} - \frac{1}{\sqrt{2\pi}} \frac{1}{(1+iw)^2} + \frac{1}{\sqrt{2\pi}} \frac{1}{(1+iw)^2} e^{1+iw}}
\end{aligned}$$

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$$f(x) = \begin{cases} |x| & \text{for } -1 < x < 1 \\ 0 & \text{ellers} \end{cases}$$

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \left(\int_{-1}^{0} -x e^{-iwx} dx + \int_{0}^{1} x e^{-iwx} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} x e^{-iwx} \Big|_{-1}^{0} - \int_{-1}^{0} \frac{1}{iw} e^{-iwx} dx - \frac{1}{iw} x e^{-iwx} \Big|_{0}^{1} + \int_{0}^{1} \frac{1}{iw} e^{-iwx} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} e^{iw} + \frac{1}{(iw)^2} e^{-iwx} \Big|_{-1}^{0} - \frac{1}{iw} e^{-iw} - \frac{1}{(iw)^2} e^{-iwx} \Big|_{0}^{1} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} e^{iw} - \frac{1}{w^2} + \frac{1}{w^2} e^{iw} - \frac{1}{iw} e^{-iw} + \frac{1}{w^2} e^{-iw} - \frac{1}{w^2} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(-\frac{2}{w^2} + \frac{2}{w^2} w \sin w + \frac{2}{w^2} \cos w \right) \\
&= \frac{\sqrt{2}}{\sqrt{\pi} w^2} \left(\cos w + w \sin w - 1 \right)
\end{aligned}$$

Brukte at $e^{iw} = \cos w + i \sin w$