Fra Kreyszig (10th), avsnitt 13.1

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$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{-2-5i}{3+i}\frac{3-i}{3-i} = \frac{1}{10}(-11-13i)$$
$$\frac{z_1}{z_2} = \frac{-2+5i}{3-i}\frac{3+i}{3+i} = \frac{1}{10}(-11+13i)$$
$$\Rightarrow \overline{z_1/z_2} = \frac{1}{10}(-11-13i)$$

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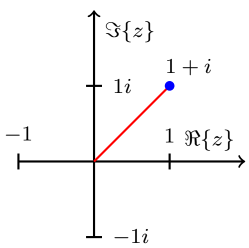
$$\frac{1}{z} = \frac{\overline{z}}{z\cdot\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$$
$$\implies \operatorname{Im}\frac{1}{z} = \frac{-y}{x^2+y^2}$$

$$z^2 = (x+iy)^2 = x^2+2ixy+i^2y^2 = (x^2-y^2)+i2xy$$
$$\frac{1}{z^2} = \frac{\overline{z^2}}{z^2(\overline{z^2})} = \frac{(x^2-y^2)-i2xy}{(x^2-y^2)^2+4x^2y^2}$$
$$\implies \operatorname{Im}\frac{1}{z^2} = \frac{-2xy}{(x^2+y^2)^2}$$

Fra Kreyszig (10th), avsnitt 13.2

1 Vi har $r = |z| = \sqrt{1^2+1^2} = \sqrt{2}$ og $\theta = \pi/4$. Dette gir

$$1+i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4).$$

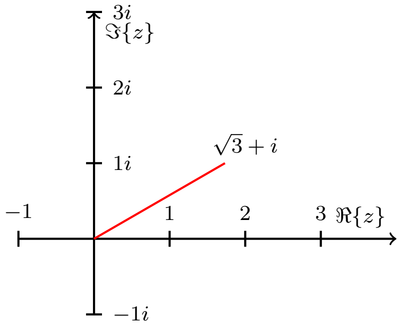


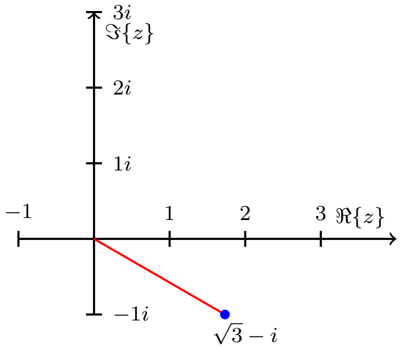
11 Tallet $\sqrt{3}+i$ ligger i første kvadrant, så

$$\operatorname{Arg}(\sqrt{3}+i) = \arctan 1/\sqrt{3} = \pi/6.$$

Ved symmetri er

$$\operatorname{Arg}(\sqrt{3}-i) = -\pi/6$$





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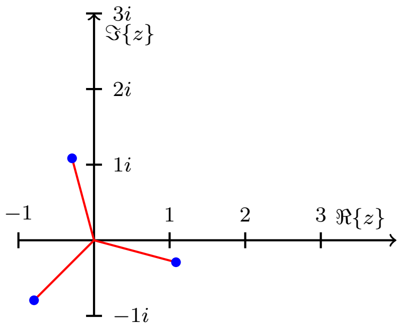
$$|1-i| = \sqrt{1^2+1^2} = \sqrt{2}, \quad \arg(1-i) = \arctan\frac{-1}{1} = -\frac{\pi}{4}$$
$$\implies 1-i = \sqrt{2}e^{i\left(-\frac{\pi}{4}+n\cdot 2\pi\right)}, \quad n \in \mathbb{Z}$$

La $w = \sqrt[3]{1-i} = Re^{i\phi}$

$$\begin{aligned} w^3 &= 1-i \\ \iff R^3e^{i3\phi} &= \sqrt{2}e^{i\left(-\frac{\pi}{4}+n\cdot 2\pi\right)} \\ \implies R^3 &= \sqrt{2}, \quad 3\phi = -\frac{\pi}{4}+n\cdot 2\pi \\ \implies w &= 2^{\frac{1}{6}}e^{i\left(-\frac{\pi}{12}+n\cdot\frac{2}{3}\pi\right)}, \quad n \in \mathbb{Z} \end{aligned}$$

Dermed finnes det tre ulike røtter. F.eks med $n = 0, 1, 2$:

$$\sqrt[3]{1-i} = \{2^{\frac{1}{6}}e^{-i\frac{\pi}{12}}, \quad 2^{\frac{1}{6}}e^{i\frac{7\pi}{12}}, \quad 2^{\frac{1}{6}}e^{i\frac{15\pi}{12}}\}$$



Fra Kreyszig (10th), avsnitt 13.3

6 Siden

$$\frac{1}{z} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

er

$$1 > \mathrm{Re}\frac{1}{z} = \frac{x}{x^2+y^2}$$

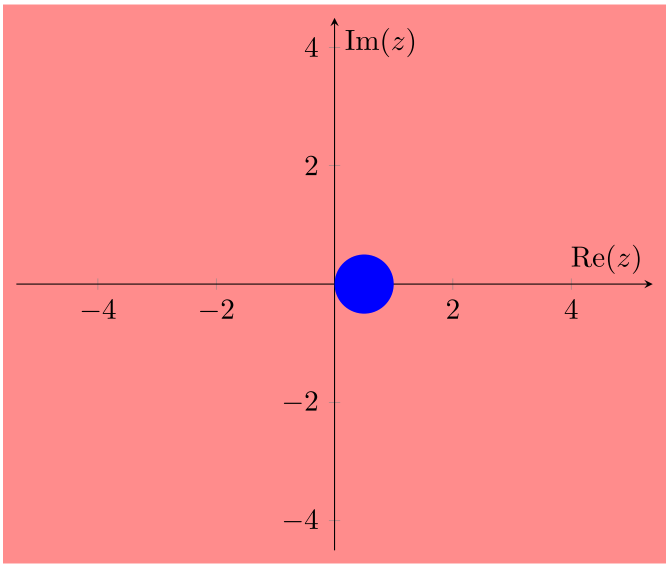
eller ekvivalent

$$x^2 + y^2 - x > 0$$

Fullfører kvadratene og får

$$(x - 1/2)^2 + y^2 > (1/2)^2$$

Dvs. $\mathrm{Re}(1/z) < 1$ er komplementet til en lukket disk med sentrum $1/2 + 0i$ og radius $1/2$. Altså det rød områådet, alt som ligger utenfor sirkelen i bildet nedenfor.



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$$f(z) = |z|^2\, \mathrm{Im}\left(\frac{1}{z}\right) \quad z \neq 0, \qquad f(0) = 0$$

For at $f(z)$ skal være kontinuerlig i punktet $z = 0$ må

$$\lim_{z\to 0} f(z) = f(0)$$

Skriver om $f(z)$ med $z = x + yi$:

$$\begin{aligned} f(z) &= \left(x^2+y^2\right)\text{Im}\left(\frac{1}{x+yi}\right) \\ &= \left(x^2+y^2\right)\text{Im}\left(\frac{x-yi}{x^2+y^2}\right) \\ &= \left(x^2+y^2\right)\left(\frac{-y}{x^2+y^2}\right) \\ &= -y \end{aligned}$$

$$\begin{aligned} => \quad \lim_{z\to 0} f(z) &= \lim_{(x,y)\to(0,0)}(-y) \\ &= 0 \end{aligned}$$

Dermed er $f(z)$ kontinuerlig i punktet $z = 0$.

Alternativ metode med (r, θ):

$$z = r\mathrm{e}^{\theta i}$$

$$\begin{aligned} => \quad f(z) &= r^2\text{Im}\left(\frac{1}{r\mathrm{e}^{\theta i}}\right) \\ &= r\text{Im}\left(\mathrm{e}^{-\theta i}\right) \\ &= r\text{Im}\left(\cos(-\theta) + i\sin(-\theta)\right) \\ &= -r\sin\theta \end{aligned}$$

Som gir

$$\begin{aligned} \lim_{z\to 0} f(z) &= \lim_{r\to 0}(-r\sin\theta) \\ &= 0 \end{aligned}$$

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$$f(z) = \frac{z-i}{z+i}$$

Deriverer på vanlig måte:

$$\begin{aligned} f'(z) &= \frac{1\cdot(z+i) - (z-i)\cdot 1}{(z+i)^2} \\ &= \frac{2i}{(z+i)^2} \end{aligned}$$

$$\begin{aligned} => \quad f'(i) &= \frac{2i}{(2i)^2} \\ &= \frac{1}{2i} = -\frac{1}{2}i \end{aligned}$$

Alternativ metode: Kan bruke definisjonen av den deriverte:

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(i + \Delta z - i)/(i + \Delta z + i) - 0}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{1}{2i + \Delta z} \\ &= \frac{1}{2i} = -\frac{1}{2} i \end{aligned}$$