NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4120 Matematikk 4K**

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Originalen er:	
1-sidig ☐	**2-sidig** ☒
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Date Signature

Problem 1

a) Use Laplace transform to compute the convolution product $\cosh t \star \sin t$ for $t \geq 0$.

b) Solve the initial value problem

$$\begin{cases} y' + y \star \cos t = 0\,, \\ y(0) = 1\,. \end{cases}$$

Solution:

a) We have that

$$\mathcal{L}[\cosh t \star \sin t] = \mathcal{L}[\cosh t]\mathcal{L}[\sin t] = \frac{s}{s^2-1}\frac{1}{s^2+1} = \frac{1/2\,s}{s^2-1} - \frac{1/2\,s}{s^2+1} = \frac{1}{2}\mathcal{L}[\cosh t] - \frac{1}{2}\mathcal{L}[\cos t]\,,$$

so $\cosh t \star \sin t = \frac{1}{2}(\cosh t - \cos t)$.

grading: +3 points if the partial fraction expansion is done right, +2 if the inverse is done correct.

b) We have that

$$\mathcal{L}[y' + y \star \cos t] = \mathcal{L}[y'] + \mathcal{L}[y \star \cos t] = s\mathcal{L}[y] - y(0) + \mathcal{L}[y]\mathcal{L}[\cos t] = sY - 1 + Y\frac{s}{s^2+1} = \mathcal{L}[0] = 0\,,$$

where $Y = \mathcal{L}[y]$. Therefore,

$$sY + \frac{sY}{s^2+1} = 1 \Rightarrow sY(1 + \frac{1}{s^2+1}) = sY\frac{s^2+2}{s^2+1} = 1 \Rightarrow Y = \frac{s^2+1}{s(s^2+2)} = \frac{1}{s}\frac{s^2+2-1}{s^2+2}$$

$$Y = \frac{1}{s}(1 - \frac{1}{s^2+2}) = \frac{1}{s} - \frac{1}{s(s^2+2)} = \frac{1}{s} - \frac{1/2}{s} + \frac{(1/2)s}{s^2+2} = \frac{1/2}{s} + \frac{(1/2)s}{s^2+2}$$

$$Y = \mathcal{L}[\frac{1}{2}] + \frac{1}{2}\mathcal{L}[\cos\sqrt{2}t] \Rightarrow y = \frac{1}{2} + \frac{1}{2}\cos\sqrt{2}t\,.$$

grading: +2 if the conversion of the diff, equation to Laplace transform is done correctly, +2 if Y is isolated correctly and the partial fraction expansion is done correctly, +1 if the inverse is correct.

Problem 2 Find the Fourier series of the 2π-periodic even expansion of the function $f(x) = e^x$ for $t \in [0, \pi]$.

Solution: Let $f_{even}(x)$ be the 2π-periodic even extension of $f(x)$. Then we have that the coefficients $b_n = 0$ for every $n \geq 1$, and the coefficients

$$a_0 = \frac{1}{\pi} int_0^\pi e^x \, dx = \frac{1}{\pi}[e^x]_{x=0}^{x=\pi} = \frac{1}{\pi}(e^\pi - 1)\,,$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi e^x \cos nx \, dx = \frac{2}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n \sin nx)\right]_{x=0}^{x=\pi} = \frac{2}{\pi}\left(\frac{e^\pi}{1+n^2}\cos n\pi - \frac{1}{1+n^2}\right)$$

$$a_n = \frac{2}{\pi(1+n^2)}((-1)^n e^\pi - 1)\,.$$

grading: +2 points if it is using the correct formulas for the even expansion + 2 if the coeficient a_0 is computed correctly + 6 if the coefficients a_n are computed correctly.

Problem 3

1. Find all the solutions of the partial differential equation

$$u_{xx}(x,t) = 2u_{tt}(x,t) + 2u_t(x,t) \qquad \text{for } x \in [0,\pi] \text{ and } t \geq 0\,, \qquad (1)$$

of the form $u(x,t) = F(x)G(t)$ that satisfies the border conditions

$$u(0,t) = 0 = u(\pi,t) \qquad \text{for } t \geq 0\,.$$

2. Find a solution from (1) that satisfies the initial conditions

$$u(x,0) = sin(2x) \qquad \text{and} \qquad u_t(x,0) = 0\,.$$

Solution:

1) We want to find soltions of the form $u(x,t) = F(x)G(t)$, then the differential equation can be written as

$$F''G = 2FG'' + FG' \Rightarrow \frac{F''}{F} = \frac{2G'' + 2G'}{G}\,,$$

therefore

$$\frac{F''}{F} = k \Rightarrow F'' - kF = 0 \qquad \text{and} \qquad \frac{2G'' + 2G'}{G} = k \Rightarrow 2G'' + 2G' - kG = 0$$

for some constant $k \in \mathbb{R}$.

We first can suppose that **k=0**: But then $F'' = 0$, so $F(x) = Ax + B$. But since the border condition implies that $F(0) = 0$ and $F(\pi) = 0$ it follows that $F(x) = 0$.

Now we can suppose that $\underline{k = p^2 > 0}$: Then $F'' - p^2F = 0$, so the solutions are of the form $F(x) = Ae^{px} + Be^{-px}$. Again because $F(0) = 0 = F(\pi)$ it follows that $F(x) = 0$.

Finally we can suppose that $\underline{k = -p^2 < 0}$: Then $F'' + p^2F = 0$, so the solutions are of the form $F(x) = A\cos px + B\sin px$. Using the border conditions we have that $A = 0$ and that $\sin p\pi = 0$, therefore $p = 1, 2, 3, \ldots$. We then denote the solution $F_n(x) = \sin nx$.

Now let $k = -p^2 = -n^2$ for some $n = 1, 2, 3, \ldots$, then $2G'' + 2G' + n^2G = 0$, so the solutions are of the form $G_n(t) = e^{\frac{-t}{2}}(A_n \cos(\frac{\sqrt{2n^2-1}}{2}t) + B_n \sin(\frac{\sqrt{2n^2-1}}{2}t))$.

Thus, we have that

$$u_n(x,t) = F_n(x)G_n(t) = \sin(nx)\, e^{\frac{-t}{2}}(A_n \cos(\frac{\sqrt{2n^2-1}}{2}t) + B_n \sin(\frac{\sqrt{2n^2-1}}{2}t))\,,$$

so $u(x, y) = \sum_{n=1}^{\infty} u_n(x, t)$ for some $A_n, B_n \in \mathbb{R}$ that makes it converge.

grading: +1 if the two differential equations are computed correctly +2 if the solutions of $F(x)$ are computed correctly and +2 if the $G(t)$ is computed correctly. and +1 if the general solution is formed.

2) Now suppose that

$$\sin 2x = u(x,0) = \sum_{n=1}^{\infty} u_n(x,0) = \sum_{n=1}^{\infty} F_n(x)G_n(0) = \sum_{n=1}^{\infty} \sin(nx)\, A_n\,,$$

therefore $A_2 = 1$ and $A_n = 0$ if $n \neq 2$.

Now let

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(nx)\,(\frac{-1}{2}e^{\frac{-t}{2}}(A_n \cos(\frac{\sqrt{2n^2-1}}{2}t) + B_n \sin(\frac{\sqrt{2n^2-1}}{2}t)) +$$

$$+ e^{\frac{-t}{2}}(-A_n\frac{\sqrt{2n^2-1}}{2} \sin(\frac{\sqrt{2n^2-1}}{2}t) + B_n\frac{\sqrt{2n^2-1}}{2} \cos(\frac{\sqrt{2n^2-1}}{2}t)))\,,$$

so

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin(nx) \left(\frac{-1}{2} A_n + B_n \frac{\sqrt{2n^2-1}}{2} \right) = 0\,,$$

Therefore, $\left(\frac{-1}{2} A_n + B_n \frac{\sqrt{2n^2-1}}{2}\right) = 0$ for every $n = 1, 2, 3, \ldots$.

But $\frac{-1}{2} 1 + B_2 \frac{\sqrt{7}}{2} = 0$, so $B_2 = \frac{1}{\sqrt{7}}$, and $B_n \frac{\sqrt{2n^2-1}}{2} = 0$ when $n \neq 2$, so $B_n = 0$.

Then the final solution is of the form

$$u(x,t) = \sin 2x e^{\frac{-t}{2}} \left(\cos(\frac{\sqrt{7}}{2} t) + \frac{1}{\sqrt{7}} \sin(\frac{\sqrt{7}}{2} t) \right)$$

grading: +2 if it is found the solutions such that $u(x,0) = \sin 2x$ and +2 if it is found correctly the final solution.

Problem 4

1. State the Cauchy-Riemann equations.
2. Determine a function $u(x,y)$ such that the function

$$f(x+iy) = u(x,y) + i(x^2 - y^2 + x + 1)$$

   is analytic in $\mathbb{C}$.

Solution:

1) An analytic function $f(x+iy) = u(x,y) + iv(x,y)$ must satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

grading: +2 for each of the equations written correctly

2) We have that $v(x,y) = x^2 - y^2 + x + 1$, therefore we have that

$$\frac{\partial v}{\partial y} = -2y = \frac{\partial u}{\partial x} \Rightarrow u = -2yx + k(y)\,.$$

On the other hand

$$\frac{\partial u}{\partial y} = -2x + k'(y) = -\frac{\partial v}{\partial x} = -2x-1 \Rightarrow -2x+k'(y) = -2x-1 \Rightarrow k'(y) = -1 \Rightarrow k(y) = -y+C\,.$$

Therefore $u(x, y) = -2xy - y + C$ for any $C \in \mathbb{R}$.

grading: +3 if the first integral is done correctly and +3 if the final solution is computed correctly

Problem 5

1. Find and classify all the singularities of the function

$$f(z) = e^{1/z} + \frac{z}{z^2-1}\,.$$

2. Find the two Laurent series of $f(z)$ with center $z_0 = 0$

Solution:

1) The singularities of $f(z)$ are the points $z = 1, -1, 0$. The singularites $z = 1, -1$ are both of order 1 and $z = 0$ is an essential singularity.

grading: +1 for each correct singularity and order. +1 for a good justification of the orders.

2) The Laurent serie of $e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}$ for $z > 0$. Now observe that $\frac{z}{z^2-1} = \frac{1/2}{z-1} + \frac{1/2}{z+1}$, so

$$\frac{1/2}{z-1} = -\frac{1}{2}\sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1$$

and

$$\frac{1/2}{z+1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n \qquad \text{for } |z| < 1$$

and on the other hand

$$\frac{1/2}{z-1} = \frac{1}{2}\frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{2}\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{z^n} \qquad \text{for } |z| > 1$$

and

$$\frac{1/2}{z+1} = \frac{1}{2}\frac{1}{z}\frac{1}{1+\frac{1}{z}} = \frac{1}{2}\frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n} \qquad \text{for } |z|>1\,.$$

Therefore the 2 Laurent series of $f(z)$ are

$$\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n \qquad \text{for } |z|<1$$

and

$$\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n} \qquad \text{for } |z|>1\,.$$

grading: +3 for each of the correct Laurent series of the function

Problem 6

1. Compute the integral
$$\oint_{\mathcal{C}} \frac{z+1}{z^3-2z^2}\,dz$$
where $\mathcal{C} = \{e^{i\theta} : t \in [0, 2\pi]\}$.
2. Compute the principal value of the integral
$$\int_{-\infty}^{\infty} \frac{x+1}{x^3-2x^2}\,dx$$

Solution:

1) First observe that $\oint_{\mathcal{C}} \frac{z+1}{z^3-2z^2}\,dz$ is equal to $2\pi i$ times the sum of all the residues of the singularities closed by the closed path $\mathcal{C}$. In this case the function $\frac{z+1}{z^3-2z^2}$ has two singularities $z=0$ and $z=2$, therefore

$$\oint_{\mathcal{C}} \frac{z+1}{z^3-2z^2}\,dz = 2\pi i \mathrm{Res}_{z=0}\left(\frac{z+1}{z^3-2z^2}\right)$$

Now one can check that $z=0$ is a singularity of order 2, so

$$\text{Res}_{z=0}\left(\frac{z+1}{z^3-2z^2}\right)=\lim_{x\to 0}\left(z^2\frac{z+1}{z^3-2z^2}\right)'=\lim_{x\to 0}\left(\frac{z+1}{z-2}\right)'=\lim_{x\to 0}\frac{-3}{(z-2)^2}=-\frac{3}{4}\,.$$

Therefore,

$$\oint_{\mathcal{C}}\frac{z+1}{z^3-2z^2}\,dz=2\pi i\frac{-3}{4}=-\frac{3\pi}{2}i\,.$$

grading: +2 for writting correctly the integration formula +3 for computing correctly the residue and +1 for finding the correct solution.

2) There is a mistake in this exercise so we will give full score to everybody .

grading: We give +4 to every exam,

Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero except at $t = a$, $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, and $\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$ for any continuous function g.
- **Convolution**

  For functions defined on the real line:

  $$f * g\,(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}.$$

  For functions defined only on the positive half-axis:

  $$f * g\,(x) = \int_{0}^{x} f(y)g(x-y)dy, \quad x > 0.$$

Laplace transform

- Definition: $\mathcal{L}[f](s) = F(s) = \int_0^{\infty} f(t)e^{-st}dt$

General formulas		$f(t)$	$F(s)$
		1	$\frac{1}{s}$
$\mathcal{L}[e^{at}f(t)](s) = F(s-a)$		$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
$\mathcal{L}[f'](s) = s\mathcal{L}[f] - f(0)$		e^{at}	$\frac{1}{s-a}$
$\mathcal{L}[f''](s) = s^2\mathcal{L}[f] - sf(0) - f'(0)$		$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}\left[\int_0^t f(\tau)d\tau)\right](s) = \frac{1}{s}\mathcal{L}[f]$		$\cos bt$	$\frac{s}{s^2+b^2}$
$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$		$\sin bt$	$\frac{b}{s^2+b^2}$
$\mathcal{L}[f(t-c)u(t-c)](s) = e^{-cs}F(s),\ c > 0$		$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{L}[tf(t)](s) = -F'(s)$		$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty F(\sigma)d\sigma$		$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
		$\delta(t-c), c > 0$	e^{-cs}
		$\cosh bt$	$\frac{s}{s^2-b^2}$
		$\sinh bt$	$\frac{b}{s^2+b^2}$

Fourier series and Fourier transform

- $2L$-periodic functions, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx$$

- Functions defined on the whole real line (need not be periodic)

$$\hat{f}(w) = \mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx,$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw.$$

- Parseval's identities

$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2, \qquad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw$$

General formulas		$f(x)$	$\hat{f}(w)$
$\widehat{f'(x)} = iw\hat{f}(w)$		$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}} e^{-iaw}$
$\widehat{f''(x)} = -w^2 \hat{f}(w)$		$\begin{cases} 1, & -b \le x \le b \\ 0, & \|x\| > b \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin bw}{w}$
$\widehat{f(x-a)} = e^{-iaw} \hat{f}(w)$		$e^{-ax} u(x)$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\hat{f}(w-b) = \widehat{e^{ibx} f(x)}$		$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|w\|}}{a}$
$\widehat{f*g} = \sqrt{2\pi} \hat{f} \hat{g}$		e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i \sin y)$
- $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}, \qquad \cosh z = \frac{e^z+e^{-z}}{2}, \quad \sinh z = \frac{e^z-e^{-z}}{2}$
- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad b_n = \frac{1}{2\pi i}\oint_C f(z)(z-z_0)^{n-1} dz$$

Some useful integrals

$\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$

$\int x \cos ax\, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$

$\int x^2 \sin ax\, dx = \frac{2}{a^2} x \sin ax + \frac{2-a^2x^2}{a^3} \cos ax + C$

$\int x^2 \cos ax\, dx = \frac{2}{a^2} x \cos ax - \frac{2-a^2x^2}{a^3} \sin ax + C$

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$

$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \ a > 0$

Some trigonometric identities

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

Some important series

- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, $\quad \sum_{n=0}^{\infty} x^n$ diverges for $|x| \geq 1$.
- $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for $x \in \mathbb{R}$.

Linear second order differential equations

Let r_1 and r_2 solve $r^2 + ar + b = 0$. Then

$$y''(x) + ay'(x) + by = 0$$

has general solution given by:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad \text{if} \quad r_1 \neq r_2, \quad r_1, r_2 \in \mathbb{R},$$

$$y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x} \quad \text{if} \quad r_1 = r_2, \quad r_1, r_2 \in \mathbb{R},$$

$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \quad \text{if} \quad r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta, \quad \alpha, \beta \in \mathbb{R}.$$