8-9-2025

Mat 4K

I dag: 10.4, Folland 2.3, konv.-nobat

A. Bessel og Parseval $(p = 2\pi)$

$$2a_0^2 + \sum_{n=1}^{k}(a_n^2 + b_n^2) \overset{\text{Tm.1 sist}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx - \underbrace{\|f - S_{f,k}\|^2}_{\geq 0} \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx$$

$$\Downarrow \; k \to \infty$$

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx$$ Bessels ulikhet

Tm. 1: Parsevals identitet

Hvis $\int_{-\pi}^{\pi} f^2\,dx < \infty$, da er

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \left(\overset{\text{s.k}}{=} 2\sum_{n=-\infty}^{\infty} |c_n|^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx$$

[$\leq$: Bessel, $\geq$: Lin. met. (3. kl.)]

Divergens-testen gir da:

Lem. 1: Riemann-Lebesgue

$$\int_{-\pi}^{\pi} f^2\,dx < \infty \implies a_n^2 + b_n^2 = 2|c_n|^2 \xrightarrow[n\to\infty]{} 0$$

Eks. 1: Summere rk.

$$S = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2n+1)^2} + \dots = ?$$

Bruk $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$, 2π-per.

$$S_f(x) \overset{\text{tidl.}}{=} \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right),$$

og Parseval:

$$2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f^2}_{=1} dx = \left(\frac{4}{\pi}\right)^2 \underbrace{\left(1^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{5}\right)^2 + \cdots\right)}_{S}$$

$$\Rightarrow \quad S = \frac{\pi^2}{8}$$

B. Konvergens av F.-rk'er $(p = 2\pi)$

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \overset{\text{def}}{=} \lim_{N\to\infty} \underbrace{\sum_{n=-N}^{N} c_n e^{inx}}_{=: S_{f,N}(x)}$$

Tm. 2: Konv. og sum

Hvis $f(x)$ (2π-per.) stk.vis kont. og $f'(a)$ eksisterer,

da konv. S_f i $x = a$ og $S_f(a) = f(a)$.

Obs. 1: $f'(a)$ eks. $\Rightarrow f(a^+) = f(a^-) = f(a)$, $\frac{d^\pm}{dx} f(a) = f'(a)$

Tidl. resultat (u. bevis)

$\Rightarrow \quad S_f(a) = \frac{1}{2}(f(a^+) + f(a^-)) = f(a)$

f stk.vis kont.

Bevis: Bruk slides!

1.) $a = 0 = f(a)$:

La $\tilde{f}(x) := \dfrac{f(x)}{e^{ix}-1}$, $x \neq 0$; $\tilde{f}(0) = \frac{1}{i} f'(0)$

Obs 1: $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\left(e^{ix}-1\right) e^{-inx}\,dx$$

$$= \tilde{c}_{n-1} - \tilde{c}_n$$

$$\Rightarrow S_{f,N}(0) \underset{x=0}{=} \sum_{n=-N}^{N} c_n \cdot \underbrace{e^{in\cdot 0}}_{=1} = \sum_{n=-N}^{N} \left(\tilde{c}_{n-1} - \tilde{c}_n\right)$$

$$= \tilde{c}_{-N-1} - \tilde{c}_N$$

Obs 2: $\tilde{f}$ stk. vis kont. i $[-\pi, \pi]$

$\left[\tilde{f}(x) \underset{x \text{ liten}}{\overset{\text{Taylor}}{\simeq}} \dfrac{f(x)}{ix} \xrightarrow[x\to 0]{} \frac{1}{i} f'(0) = \tilde{f}(0)\right.$, $x \neq 0$ ok. $\left.\right]$

$$\Rightarrow \int_{-\pi}^{\pi} |\tilde{f}(x)|^2\,dx \leq 2\pi \max_{x\in[-\pi,\pi]} |\tilde{f}(x)|^2 < \infty$$

$$\Rightarrow S_{f,N}(0) = \tilde{c}_{-N-1} - \tilde{c}_N \xrightarrow[\text{Lem. 1}]{\text{R.-L.}} 0 - 0 = f(0)$$

2.) Generell a, $f(a)$:

La $g(x) = f(x+a) - f(a)$.

Obs.: $g(0) = 0$, $g'(0) = f'(a)$, $S_{g,N}(x) \overset{\text{sjk.}}{=} S_{f,N}(x+a) - f(a)$

$$\Rightarrow |S_{f,N}(a) - f(a)| = |S_{g,N}(0)| \xrightarrow[N\to\infty]{1.)} 0 \qquad \square$$

C. Uniform konv. og deriverbarhet

$S(x) = \sum_{n=-\infty}^{\infty} g_n(x)$, $S_N(x) = \sum_{n=-N}^{N} g_n(x)$ [g_n kont.] [kont. på $[-\pi,\pi]$]

$S(x)$ konv. abs. hvis $\sum_{n=-\infty}^{\infty} |g_n(x)|$ konv.

$S(x)$ konv. uniformt på $I \subset \mathbb{R}$ lukket int. hvis

$$\underset{x \in I}{\max} |S(x) - S_N(x)| \xrightarrow[N \to \infty]{} 0$$

Lem. 2: S_N kont., $S_n \xrightarrow{\text{unif.}} S$ $\Rightarrow$ S er kont.

$$\Big[\, |S(x) - S(y)| \le \underbrace{|S(x) - S_N(x)|}_{\substack{\max \to 0 \\ x \;\; N \to \infty}} + \underbrace{|S_N(x) - S_N(y)|}_{\substack{\to 0 \\ y \to x}} + \underbrace{|S_N(y) - S(y)|}_{\substack{\max \to 0 \\ y \;\; N \to \infty}} \xrightarrow{y \to x} 0 \,\Big]$$

(unif. konv. — kont. — unif. konv.)

Lem. 2:

Weierstrass M-test:

$|g_n(x)| \le M_n$ for alle $x \in I$ (lukket int.) og $\sum_{n=-\infty}^{\infty} M_n < \infty$

$\Downarrow$

$S(x)$ konv. abs. og unif. i I

$$\Big[\, \underset{x}{\text{"max"}} |S(x) - S_N(x)| \le \underset{x}{\text{"max"}} \sum_{|n|>N} |g_n(x)| \le \sum_{|n|>N} M_n \xrightarrow[N \to \infty]{} 0 \,\Big]$$

"feil"

Obs. 2:

$g_n(x) = a_n \cos nx \Rightarrow |g_n| \le |a_n|$

$g_n(x) = c_n e^{inx} \Rightarrow |g_n| \le |c_n|$

osv.

$|c_n| \le |a_n| + |b_n| \le 2|c_n|$ $\left(c_{\pm n} = \frac{1}{2}(a_n \mp i b_n) \right)$ $\le |c_n| + |c_{-n}|$

Tm. 3: Hvis $f(x)$ 2π-per., kont., f' stk.v. kont., da konv. S_f til f abs. og unif. i $[-\pi,\pi]$

Bevis:

$$S_f = \sum_n c_n e^{inx}, \quad S_{f'} \overset{\text{tidl.}}{=} \sum \underbrace{in c_n}_{c_n'} e^{inx}$$

$$\Longrightarrow \left[\sum_n |c_n'|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(x)|^2 dx < \infty \right.$$

Bessel $\qquad$ f' stk. v. kont.

$$\sum_n |c_n| = |c_0| + \sum_{n\neq 0} \left|\frac{c_n'}{n}\right| \le |c_0| + \underbrace{\Big(\sum_{n\neq 0} \frac{1}{n^2}\Big)^{\frac12}}_{<\infty} \underbrace{\Big(\sum_{n \neq 0} |c_n'|^2\Big)^{\frac12}}_{<\infty} < \infty$$

C.-S.: $|\vec{a}\cdot\vec{b}| \le |\vec{a}|\cdot|\vec{b}|$

$$\sum_{n=-N}^{N} a_n b_n \le \Big(\sum_{-N}^{N} a_n^2\Big)^{\frac12} \Big(\sum_{-N}^{N} b_n^2\Big)^{\frac12}, \quad N \to \infty$$

Weierstrass M-test $\underset{g_n = c_n e^{inx}}{\overset{M_n = |c_n|}{\Longrightarrow}}$ S_f abs., unif. konv. (i $[-\pi,\pi]$)

~~Tidl.~~ Tm. 1 (konv.) ~~fm~~ $\Longrightarrow$ $S_f = f$ (i $[-\pi,\pi]$) $\square$

Jo raskere $2|c_n| = |a_n| + |b_n|$ avtar, jo flere deriverte har f:

Tm. 4: f 2π-per., $k \in \{0,1,2,\dots\}$

a) $f^{(k-1)}$ kont., $f^{(k)}$ stk. v. kont. $\Longrightarrow$ $n^k a_n, n^k b_n, n^k c_n \to 0$

b) $|c_n| \overset{n\neq 0}{\le} C|n|^{-k-\alpha}$ $\big(\overset{\text{Obs.2}}{\Longleftrightarrow} |a_n| + |b_n| \le C n^{-k-\alpha}\big)$, $\alpha > 1$

$\Longrightarrow$ $f^{(k)}$ eks. og er kont. og $f^{(k)} = S_{f^{(k)}}$

Bevis:

a) $c_n^{(k)} = (in)^k c_n \to 0$ by Lem. 1 (R.-L.) siden $f^{(k)}$ stk. v. kont.

b) $\displaystyle \sum_{n\neq 0} |n^k c_n| \le C \sum_{n \neq 0} |n|^k |n|^{-k-\alpha} = C \sum_{n\neq 0} |n|^{-\alpha} \overset{\alpha>1}{<} \infty$ Mat. 1 (integral test)

$\underset{\text{Weierstrass M-test}}{\Longrightarrow}$ $S_{f^{(k)}}(x) = \sum (in)^k c_n e^{inx}$ konv. abs. og unif.

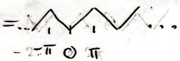
Lem. 2

$\Longrightarrow \quad S_{f^{(k)}}(x)$ kont. (Lem 2) og $= f^{(k)}(x)$ overalt $\Box$ 6.

Kan vises at $f^{(k)} = S_{f^{(k)}}$ (leddvis deriv.; $\overset{\text{Tm. 4 i,}}{\text{K 15.5}}$ (senere)) $\Box$

Fourier-rad. -"-

Eks. 2:

$f(x) =$  $(p = 2\pi)$

$$S_f(x) \overset{\text{tidl.}}{=} \frac{1}{2} - \frac{4}{\pi}\Big(\cos x + \frac{1}{3^2}\cos 3x + \cdots + \underbrace{\frac{1}{(2m+1)^2}}_{a_{2m+1}}\cos(2m+1)x + \cdots\Big)$$

Obs: $|a_n| + |b_n| \le 0 + \frac{4}{\pi}\frac{1}{n^2}$

Tm 4 b $\underset{\alpha = 2}{\overset{k=0}{\Longrightarrow}}$ S_f konv. abs. og unif., $f^{(0)} = f$ er kont.

Obs: f' eks. ikke i $x = n\pi$, $n \in \mathbb{Z}$

12:01