Mat 4K 12.9.2025

Rep.: slides

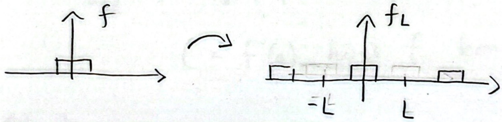
Info: www

I dag: 11.7, 11.9

A. Fourier integral

$f(x)$, $x \in \mathbb{R}$, vilkårlig

$f_L(x) := f(x)$, $x \in [-L, L]$; $\quad f_L(x+2L) = f_L(x)$



Obs: $f(x) = \lim_{L\to\infty} f_L(x)$

$$f_L(x) \underset{\substack{\uparrow \\ \text{betingelser}}}{=} S_{f_L}(x) \overset{\text{F.rk}}{=} \sum_{n=-\infty}^{\infty} \Big[\frac{1}{2L} \int_{-L}^{L} \underbrace{f_L(x)}_{=f(x)} e^{-\frac{in\pi x}{L}} dx \Big] e^{i\frac{n\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\Big[\int_{-L}^{L} f(x) e^{-i\omega_n x} dx \Big] e^{i\omega_n x} \frac{\Delta\omega}{2\pi}}_{\text{Riemann-sum!}},$$

der $\omega_n = \frac{n\pi}{L}$, $\quad \Delta\omega = \omega_n - \omega_{n-1} = \frac{\pi}{L}$.

Obs: $f(x) = \lim_{L\to\infty} f_L(x) = \lim_{L\to\infty} S_{f_L}(x) \underset{\substack{\uparrow \\ \text{må vises}}}{\overset{!}{=}} I_f(x)$

der $I_f(x)$ er <u>F.-integral</u> til $f(x)$:

$$I_f(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw,$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

<u>Tm. 1:</u> Anta

(A1) f stk. vis kont., $\int_{-\infty}^{\infty} |f|\, dx < \infty$

(A2) $\left(\frac{d}{dx}\right)^{\pm} f(a)$ eks.

Da er

$$I_f(a) = \frac{1}{2}\left(f(a^+) + f(a^-)\right)$$

$$(= f(a) \text{ hvis } f \text{ kont. i } x = a)$$

<u>Obs. 1:</u>

i) $\int_{-\infty}^{\infty} - \;=\; \lim_{a\to\infty} \int_{-a}^{0} - \;+\; \lim_{b\to\infty} \int_{0}^{b} -$

ii) Gibbs fenomen, se s. 515 i Kreyszig

<u>Eks. 1:</u> $f(x) = e^{-k|x|}$, $k > 0$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k|x|}\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{0}^{\infty} e^{-(k+iw)x}\, dx + \int_{-\infty}^{0} e^{(k-iw)x}\, dx \right)$$

3)

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{k+i\omega} e^{-(k+i\omega)x} \right]_{x=0}^{x=\infty} + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{k-i\omega} e^{(k-i\omega)x} \right]_{x=-\infty}^{x=0}$$

$$|e^{-kx-i\omega x}| = e^{-kx} |e^{-i\omega x}| = e^{-kx} \underset{x \to +\infty}{\longrightarrow} 0$$

Sjk!

$$= \frac{1}{\sqrt{2\pi}} \Big(\underbrace{\frac{1}{k+i\omega}}_{\text{"}x=0\text{"}} + \underbrace{\frac{1}{k-i\omega}}_{\text{"}x=0\text{"}} \Big) = \frac{1}{\sqrt{2\pi}} \cdot \frac{2k}{k^2+\omega^2}$$

$$I_f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} = \underline{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2k}{k^2+\omega^2} e^{i\omega x} d\omega}$$

$$\overset{\text{Tm.1}}{=} f(x) = e^{-k|x|}$$

"ensidig deriv."

4.)

B. Reell form av F. int.

$$I_f(x) \overset{(1)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \overbrace{\left[\int_{-\infty}^{\infty} f(y)\, e^{-iwy}\, dy\right]}^{\hat{f}(w)} e^{iwx}\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\underbrace{\int_{-\infty}^{\infty} f(y)\cos(wy)\, dy}_{=I_1(w),\ \text{like}} - i\underbrace{\int_{-\infty}^{\infty} f(y)\sin(wy)\, dy}_{=I_2(w),\ \text{odde}}\Big]\cdot$$

$$\cdot\,(\underbrace{\cos(wx)}_{w\text{-like}} + i\underbrace{\sin(wx)}_{w\text{-odde}})\, dw$$

(2)

$$\overset{S.\,i.\,k}{=} \boxed{\begin{aligned}&\frac{1}{2\pi}\,2\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(y)\cos(wy)\, dy\right]\cos(wx)\, dw - i\cdot 0 \\ &+ i\cdot 0 + \frac{1}{2\pi}\cdot 2\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(y)\sin(wy)\, dy\right]\sin(wx)\, dw\end{aligned}}$$

Eks. 2: $f = e^{-k|x|}$

$$I_f(x) \overset{\text{Eks.1}}{=} \frac{k}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{1}{k^2+w^2}}_{\text{like}}(\underbrace{\cos(wx)}_{\text{like}} + i\underbrace{\sin wx}_{\text{odde}})\, dw$$

$$= \frac{2k}{\pi}\int_0^{\infty}\underbrace{\frac{1}{k^2+w^2}\cos(wx)}_{\text{like}}\, dw + \underbrace{0}_{\text{odde}}$$

Obs. 2:

a) Bruk av F. int. → best. int.:

Eks.: $\displaystyle\int_0^{\infty}\frac{\cos(wx)}{k^2+w^2}\, dw \overset{\text{Eks. 2}}{=} \frac{\pi}{2k}\, e^{-k|x|} \quad (k>0, x\in\mathbb{R})$

Laplace-int.

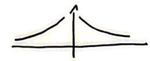
"F-cos. int."

b) f like $\overset{(2)}{\Longrightarrow}$ $\displaystyle I_f(x) = \frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(y)\cos(wy)\, dy\right]\cos(wx)\, dw$

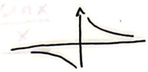
c) f odde $\overset{(2)}{\Rightarrow}$ $f(x) = \frac{2}{\pi}\int_0^\infty [\int_0^\infty \underset{=}{f(y)} \underbrace{\sin(wy)}_{\text{like}}\, dy] \sin(wx)\, dw$ "F.-sin.-int."

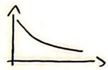
d)

$f(x),\ x \in [0,\infty)$ $\xrightarrow{\text{like utv.}}$ $f_1(x),\ x \in \mathbb{R}$



$f(x),\ x \in [0,\infty)$ $\xrightarrow{\text{odde utv.}}$ $f_2(x),\ x \in \mathbb{R}$





$x \in [0,\infty)$: $f(x) = f_1(x) = f_2(x) =$ F-cos. int. = F-sin. int.

C. Fouriertransformen

Viktig! Signalbehandling, løse diff likn ...

Likner Laplace transformen.

Def. 2:

a) F.-transf.: $\mathcal{F}[f](w) = \hat{f}(w) \overset{\text{def.}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$

b) Invers F.-tr.: $\mathcal{F}^{-1}[g](x) = \check{g}(x) \overset{\text{def.}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(w) e^{iwx}\, dw$

Obs. 3: F. int. $I_f(x) \overset{\text{Sjk}}{=} \mathcal{F}^{-1}[\mathcal{F}[f]](x)$!

Eks. 3:

i) $\mathcal{F}[e^{-k|x|}] \overset{\text{Eks. 1}}{=} \sqrt{\frac{2}{\pi}} \frac{k}{k^2 + w^2}$ $(k > 0)$

ii) $f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & \text{ellers} \end{cases}$

$\hat{f}(w) = \mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}}\int_0^a 1 \cdot e^{-iwx}\, dx \overset{\text{Sjk}}{=} \underline{\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{iw}(1 - e^{-iwa})}$

(ii) $g(\omega) = \begin{cases} 1 & |\omega|<1 \\ 0 & \text{ellers} \end{cases}$

$$\check{g}(x) = \mathcal{F}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{i\omega x}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{ix} \underbrace{(e^{ix} - e^{-ix})}_{2i \sin x} = \underline{\sqrt{\frac{2}{\pi}} \frac{\sin x}{x}}$$

<u>Obs. 4:</u> $\hat{f}, \check{g}$ kan være komplekse

<u>Tm. 2:</u>

a) (A1) (f stk. v. kont., $\int_{-\infty}^{\infty} |f|\,dx < \infty$)

$\Rightarrow \hat{f} = \mathcal{F}[f]$ og $\check{f} = \mathcal{F}^{-1}[f]$ eksisterer

b) (A1), (A2), f kont. i x

$\overset{\text{Tm. 1}}{\Longrightarrow} \mathcal{F}^{-1}[\mathcal{F}[f]](x) = f(x)$

<u>Tm. 3:</u> Egenskaper

÷| ~~a) $\hat{f}(\omega), \hat{g}(\omega)$ eks.~~ ; $a, b \in \mathbb{R}$

a) $\Rightarrow \mathcal{F}[af(x) + bg(x)](\omega) = a\mathcal{F}[f](\omega) + b\,\mathcal{F}[g](\omega)$

b) f kont., $|f(x)| \underset{|x|\to\infty}{\longrightarrow} 0$, $\int_{-\infty}^{\infty} |f'|\,dx < \infty$

$\Rightarrow \mathcal{F}[f'(x)](\omega) = i\omega\, \mathcal{F}[f](\omega)$

÷| ~~c) $\hat{f}(\omega)$ eks.~~ ; $a \in \mathbb{R}$

c) $\Rightarrow \mathcal{F}[e^{-iax} f(x)](\omega) = \mathcal{F}[f](\omega + a)$ "x-forskyvning"

"Bevis:"

b) $\int_{-\infty}^{\infty} f'(x) e^{-i\omega x} dx = \left[\cancel{f(x) e^{-i\omega x}} \right]_{x=-\infty}^{x=\infty} \nearrow 0 - (-i\omega) \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

$$|f(x) \cdot e^{-i\omega x}| = |f(x)| \underset{|x|\to\infty}{\longrightarrow} 0$$

c) $\int_{-\infty}^{\infty} e^{-iax} f(x) e^{-i\omega x} dx = \int_{-\infty}^{\infty} f(x) e^{-i(a+\omega)x} dx = \hat{f}(a+\omega)$ $\square$

Obs. 5:

a) $\mathcal{F}^{-1}[af+bg] = a\mathcal{F}^{-1}[f] + b\mathcal{F}^{-1}[g]$ "$\mathcal{F}^{-1} \sim \mathcal{F}$"

$\mathcal{F}^{-1}[f'(\omega)](x) = -ix\,\mathcal{F}^{-1}[f](x)$

$\mathcal{F}^{-1}[e^{ia\omega} f(\omega)](x) = \mathcal{F}^{-1}[f](x+a)$

b) $\mathcal{F}[f''](\omega) = i\omega\,\mathcal{F}[f'] = (i\omega)^2 \mathcal{F}[f] = -\omega^2 \mathcal{F}[f]$

c) Sml. m. Laplace transf. ; $\mathcal{F}^{-1} \sim \mathcal{F}$

Hvis tid **Eks. 4:**

~~$\vdots \mid \mathcal{F}[e^{-x^2}] \overset{\text{Tabell}}{=} \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}$~~

$\mathcal{F}[x \cdot e^{-x^2}] = \mathcal{F}[-\frac{1}{2}(e^{-x^2})'] \overset{\text{Tm. 3a}}{=} -\frac{1}{2}\mathcal{F}[(e^{-x^2})']$

$\overset{\text{Tm 3b}}{=} -\frac{i\omega}{2}\mathcal{F}[e^{-x^2}] \overset{\text{Tabell}}{=} -\frac{i\omega}{2\sqrt{2}} e^{-\frac{\omega^2}{4}}$