TMA4120 Matematikk 4K høsten 2025

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Oppgave 1 Let $f(x) = \begin{cases} 0, & x \in [-\pi, 0) \\ 2\pi, & x \in [0, \pi], \end{cases}$ be a function defined on $[-\pi, \pi]$.

1. Compute the Fourier Series of f.
2. Let $S_f(x)$ be the sum of the Fourier series of $f(x)$. Determine the following function values:

$$(a)\ S_f(0) \quad \text{and} \quad (b)\ S_f(5)$$

Solution.

1. We compute the coefficients

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}dx = \frac{1}{2\pi}\int_0^{\pi} 2\pi \cdot e^{-inx}dx = \int_0^{\pi} e^{-inx}dx = \left[\frac{1}{-in}e^{-inx}\right]_0^{\pi}$$
$$= \frac{1}{-in}(e^{-in\pi}-1) = \frac{1-\cos(\pi n)}{in} = \frac{1-(-1)^n}{in},$$
$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \int_0^{\pi} dx = \pi.$$

Thus we have

$$S_f = c_0 + \sum_{n\neq 0} c_n e^{inx} = \pi + \sum_{n\neq 0}\frac{1-(-1)^n}{in}e^{inx}$$
$$\left(= \pi + 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\cdot\frac{e^{inx}-e^{-inx}}{2i} = \pi + 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(nx)\right)$$

2. Since f (is piecewise continuous with one-sided derivatives and) has a discontinuity at 0,

$$S_f(0) = \frac{1}{2}(f(0^+)+f(0^-)) = \frac{1}{2}(2\pi+0) = \pi.$$

Since S_f is 2π periodic and f is continuous at $5-2\pi \approx -1{,}28$, we have

$$S_f(5) = S_f(5-2\pi) = f(5-2\pi) = 0.$$

Oppgave 2 Compute the line integrals $\int_C f(z)dz$ in the case

1. $f(z) = \mathrm{Im}(z^2)$ and C is the straight line segment from $z_1 = 1$ to $z_2 = 1+3i$,
2. $f(z) = 3+\cos(z)$ and C is a half circle from $z_1 = 0$ to $z_2 = i$.

Solution.

1. We take to C to be parameterized by $z(t) = 1+3it$. Then we have

$$\int_C f(z)dz = \int_0^1 \mathrm{Im}(z(t)^2)\cdot\dot{z}(t)dt = \int_0^1 \mathrm{Im}((1+3it)^2)\cdot 3i\,dt = \int_0^1 \mathrm{Im}(1+6it-9t^2)\cdot 3it\,dt$$
$$= \int_0^1 18it\,dt = 9i.$$

Thus $\mathrm{Re}(\int_C f(z)dz) = 0$ and $\mathrm{Im}(\int_C f(z)dz) = 9$.

2. Note first that f is analytic in the whole complex plane as a sum of analytic functions, so we may use the indefinite integral (antiderivative) to calculate the integral. We compute:

$$\int_C f(z)dz = \int_0^i z + 3\cos(z)\,dz = \left[\frac{z^2}{2} + 3\sin(z)\right]_0^i = -\frac{1}{2} + 3\sin(i) - 0 = -\frac{1}{2} + 3\frac{e^{i\cdot i} - e^{-i\cdot i}}{2i}$$
$$= -\frac{1}{2} + \frac{3i}{2}(e^1 - e^{-1}) = -\frac{1}{2} + 3i\sinh(1) \approx -\frac{1}{2} + i\,3.53.$$

Oppgave 3 The Laurent series $\frac{1}{(z^2-1)(z-i)} = \sum_{n=-\infty}^{\infty} a_n(z-1)^n$ converges at the point $z = \frac{5}{2}$. Determine the largest annulus $D : r < |z - z_0| < R$ where the series converges.

Solution. We see from the form of the Laurent series that $z_0 = 1$ is the center. Furthermore, the function has poles (the denominator has zeros) at $z = \pm 1$ and $z = i$. We compute the distances

$$|z_0 - 1| = |1 - 1| = 0, \quad |z_0 - (-1)| = |1 + 1| = 2, \quad |z_0 - i| = |1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

By Laurent's theorem the largest annulus of convergence D centered at $z = 1$ and containing $z = \frac{5}{2}$ is the largest annulus where f is analytic. Since $|z_0 - \frac{5}{2}| = \frac{3}{2} \in (\sqrt{2}, 2)$,

$$D = \{\sqrt{2} < |z - 1| < 2\}.$$

Thus $z_0 = 1$, $r = \sqrt{2}$, and $R = 2$.

Oppgave 4 Find a conjugate harmonic function to $u(x, y) = e^{-2y-1}\cos(2x + 1) - 3x$.

Solution. The conjugate harmonic function v must satisfy the Cauchy-Riemann equations, i.e.

$$u_x = v_y, \quad u_y = -v_x.$$

Thus we have that

$$v = \int v_x dx = -\int u_y dx = -\int (e^{-2y-21}\cos(2x+1) - 3x)_y dx = \int 2e^{-2y-1}\cos(2x+1)dx$$
$$= e^{-2y-1}\sin(2x+1) + C(y).$$

Furthermore,

$$v_y = -2e^{-2y-1}\sin(2x+1) + C'(y) \quad = \quad u_x = e^{-2y-1}(-2)\sin(2x+1) - 3.$$

Which implies $C'(y) = -3 \implies C(y) = -3y + C$ and

$$v(x, y) = e^{-2y-1}\sin(2x+1) - 3y + C.$$

Oppgave 5 Find the solution $y(t)$ of the problem

$$y' + 4y + 13\int_0^t y(\tau)d\tau = \delta(t-4), \quad y(0) = 0,$$

where δ is the Dirac delta-function.

Solution. Let $Y = \mathcal{L}[y]$. Then we get as the Laplace transform of the equation

$$sY - y(0) + 4Y + \frac{13}{s}Y = e^{-4s} \quad \Longrightarrow \quad \frac{s^2+4s+13}{s}Y = e^{-4s} \quad \Longrightarrow \quad Y = \frac{s}{s^2+4s+13}e^{-4s}.$$

By completing the square, we get $s^2 + 4s + 13 = (s+2)^2 + 3^2$. This means that we have

$$Y = \frac{s+2-2}{(s+2)^2+3^2}e^{-4s}$$

We have from the table of Laplace transforms that

$$\mathcal{L}^{-1}\left[\frac{s+2-\frac{2}{3}3}{(s+2)^2+3^2}\right] = e^{-2t}\big(\cos(3t) - \frac{2}{3}\sin(3t)\big) =: f(t).$$

Thus, by the shift theorem,

$$y = \mathcal{L}^{-1}[Y] = f(t-4)\cdot u(t-4) = e^{-2(t-4)}\Big(\cos\big(3(t-4)\big) - \frac{2}{3}\sin\big(3(t-4)\big)\Big)u(t-4)$$

where $u(t)$ is the unit step (Heaviside) function.

Oppgave 6 Show that the Fourier transform of $g(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is $\hat{g}(w) = \frac{1}{\sqrt{2\pi}}\frac{1}{1+iw}$. Then show that $u(x) = \int_{-\infty}^{\infty} g(x-y)f(y)dy$ solves the equation

$$u'(x) + u(x) = f(x) \quad \text{for} \quad x \in \mathbb{R}.$$

Solution.

$$\begin{aligned}
\mathcal{F}[g](w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-x}e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x(1+iw)}dx \\
&= \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\int_0^R e^{-x(1+iw)}dx = \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\left[\frac{-1}{1+iw}e^{-x(1+iw)}\right]_0^R \\
&= \frac{1}{\sqrt{2\pi}}\cdot\frac{-1}{1+iw}\Big(\lim_{R\to\infty} e^{-R(1+iw)} - 1\Big) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{1+iw}
\end{aligned}$$

where in the last line we used that $|e^{-R(1+iw)}| = e^{-R} \to 0$ for $R \to 0$.

To solve equaiton $u' + u = f$, we take the Fourier tranform of the equation,

$$iw\hat{u} + \hat{u} = \hat{f},$$

and solve for $\hat{u}$,

$$\hat{u} = \frac{1}{1+iw}\hat{f} = \sqrt{2\pi}\hat{g}\cdot\hat{f}.$$

We use the convolution formula to obtain

$$u = g * f = \int_{-\infty}^{\infty} g(x-y)f(y)dy.$$

Oppgave 7 Determine the value of the integral

$$\int_{-\infty}^{\infty} \frac{5-x}{(1+x^2)(4+x^2)}dx$$

Solution. First note that

$$f(z) = \frac{5-z}{(1+z^2)(4+z^2)} = \frac{5-z}{(z-i)(z+i)(z-2i)(z+2i)}.$$

Let S_R be the semicircle of radius R in the upper half plane and $C_R := [-R, R] \cup S_R$, then

$$I = \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \underbrace{\int_{-R}^{R} f(x)dx}_{} \\ = \int_{C_R} f(z)dz - \int_{S_R} f(z)dz = I_1 + I_2.$$

I_1 :

1. Singularities of $f(z)$ (0-s of the denominator): $z = \pm i, \quad z = \pm 2i,$ order 1 poles.
2. Singularities encircled by C_R: $z = i$ and $z = 2i$ in the upper half are encircled by for $R > 2$.
3. Residues:

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{5-z}{(z+i)(z-2i)(z+2i)} = \frac{5-i}{(2i)(-i)(3i)} = \frac{5-i}{6i} = \frac{-1}{6} - \frac{5i}{6},$$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i} \frac{5-z}{(1+z^2)(z+2i)} = \frac{5-2i}{(-3)(4i)} = \frac{1}{6} + \frac{5i}{12}$$

4. By the residue Theorem:

$$I_1 = 2\pi i\left(\frac{-1}{6} - \frac{5i}{6} + \frac{1}{6} + \frac{5i}{12}\right) = 2\pi i\left(\frac{-5i}{12}\right) = \frac{5\pi}{6}.$$

I_2: By ML inequality we have

$$|I_2| \leq \max_{z\in S_R} |f(z)| \cdot L = \max_{z\in S_R} |f(z)|\pi R \leq \max_{|z|=R} \frac{|5-z|}{|1+z^2| \cdot |4+z^2|}\pi R$$
$$\leq \max_{|z|=R} \frac{5+|z|}{|z|^4|1+\frac{1}{z^2}| \cdot |1+\frac{4}{z^2}|}\pi R = \frac{(5+R)\pi R}{R^4} \max_{|z|=R} \frac{1}{|1+\frac{1}{z^2}| \cdot |1+\frac{4}{z^2}|}.$$

As $R \to \infty$ we have that

$$\max_{|z|=R} \frac{1}{|1+\frac{1}{z^2}| \cdot |1+\frac{4}{z^2}|} \to 1 \qquad \text{and} \qquad \frac{(5+R)\pi R}{R^4} \to 0$$

and hence $I_2 \to 0$. We conclude that

$$I = \lim_{R\to\infty}(I_1 + I_2) = \frac{5}{6}\pi + 0 = \frac{5\pi}{6}.$$

Oppgave 8 Let $n \in \mathbb{N}$. Find a function $G_n(y)$ such that $u_n(x,y) = \sin(\frac{n\pi x}{2})G_n(y)$ solves

$$(1) \qquad \begin{cases} u_{xx} + \frac{1}{4}u_{yy} = 0, & \text{for } x \in (0,2)\, y \in (0,3), \\ u(0,y) = 0 = u(2,y), & \text{for } y \in [0,3], \\ u(x,0) = 0 & \text{for } x \in [0,2]. \end{cases}$$

Then find the solution of problem (1) and the boundary condition

$$u(x,3) = 11\sin(\pi x) + \sin(5\pi x).$$

Solution. It follows that

$$(u_n)_{xx} = -G_n(y)\left(\frac{n\pi}{2}\right)^2 \sin\left(\frac{n\pi x}{2}\right) = -\left(\frac{n\pi}{2}\right)^2 u_n,$$
$$(u_n)_{yy} = G_n''(y)\sin\left(\frac{n\pi x}{2}\right),$$

and hence

$$0 = (u_n)_{xx} + \frac{1}{4}(u_n)_{yy} = -G_n(y)\left(\frac{n\pi}{2}\right)^2 \sin\left(\frac{n\pi x}{2}\right) + \frac{1}{4}G_n''(y)\sin\left(\frac{n\pi x}{2}\right) \implies G_n'' - (\pi n)^2 G_n = 0.$$

The corrsponding characteristic equation $r^2 - (n\pi)^2 = 0$ has solutions $r = \pm n\pi$, so the general solution

$$G_n = A_n e^{n\pi y} + B_n e^{-n\pi y}.$$

The first two boundary conditions are satisfied because of the sin-term,

$$u_n(y) = G_n(y)\sin 0 = 0 = G_n(y)\sin\left(\frac{n\pi 2}{2}\right) = u_n(2,y),$$

while the last one gives:

$$0 = u_n(x,0) = G_n(0)\sin\left(\frac{n\pi x}{2}\right) \implies G_n(0) = 0 \implies 0 = A_n e^0 + B_n e^0 \implies A_n = -B_n.$$

Hence we conclude that

$$G_n(y) = A_n(e^{n\pi y} - e^{n\pi y}) = 2A_n \sinh(n\pi y).$$

To satisfy also the last boundary condition, inspired by superposition we try setting $u = \sum_{n\in\mathbb{N}} u_n$:

$$u(x,3) = 11\sin(\pi x) + \sin(5\pi x)$$
$$= \sum u_n(x,3) = \sum_{n=1}^{\infty} 2A_n \sinh(n\pi 3)\sin\left(\frac{n\pi x}{2}\right).$$

We determine A_n by comparing coefficient (the Fourier coefficients are uniques): We find that $A_n = 0$ for $n \notin \{2, 10\}$, and that

$$11 = 2A_2\sinh(6\pi) \implies 2A_2 = \frac{11}{\sinh(6\pi)},$$
$$1 = 2A_{10}\sinh(30\pi) \implies 2A_{10} = \frac{1}{\sinh(30\pi)}.$$

Hence:

$$u(x,y) = \frac{11}{\sinh(6\pi)}u_2 + \frac{1}{\sinh(30\pi)}u_{10} = 11\frac{\sinh(2\pi y)}{\sinh(6\pi)}\sin(\pi x) + \frac{\sinh(10\pi y)}{\sinh(30\pi)}\sin(5\pi x).$$

By superposition, this solution also satisfy (1) (linear and homogeneous equation and boundary conditions).

Oppgave 9 Show that $u(x,t) = \sum_{n=1}^{\infty} \frac{1}{n} e^{-n^2 t/3} \sin(nx)$ converges sufficiently uniformly and solves the equation

$$u_t - \frac{1}{3} u_{xx} = 0, \quad x \in (0, \pi),\ t > 0.$$

Solution. Assume $t \geq a > 0$. Set

$$u_n(x,t) = \frac{1}{n} e^{-n^2 t/3} \sin(nx).$$

Now we have that

$$|u_n| \leq \frac{1}{n} e^{-n^2 t/3} \leq e^{-nt/3} \leq e^{-an/3} = (e^{-a/3})^n =: M_n,$$

where $\sum M_n$ converges as a geometric series since $(e^{-a/3}) < 1$. Thus $\sum u_n$ converges uniformly by the M-test for $t \geq a$ and $x \in [0, \pi]$.

We calculate derivaties,

$$(u_n)_x = e^{-n^2 t/3} \cos(nx), \qquad (u_n)_{xx} = -n^2 u_n, \qquad \text{and} \qquad (u_n)_t = -\frac{1}{3} n^2 u_n = \frac{1}{3} (u_n)_{xx}.$$

For $t \geq a$, we observe that

$$|(u_n)_x| \leq M_n \quad \text{and} \quad |(u_n)_t| \leq |(u_n)_{xx}| \leq n^2 M_n \leq \max_{n \in \mathbb{N}} \left(n^2 e^{-an/6} \right) \cdot e^{-an/6} = C_0 (e^{-a/6})^n =: \tilde{M}_n,$$

where $C_0 := \max_{n \in \mathbb{N}} \left(n^2 e^{-an/6} \right) < \infty$ since $n^2 e^{-an/6} \to 0$. We have $\sum \tilde{M}_n < \infty$ since this series is again a geometric series and $e^{-an/6} < 1$. Again by the M-test $\sum (u_n)_x$, $\sum (u_n)_{xx}$, and $\sum (u_n)_t$ all converge uniformly for $t \geq 0$. Hence $u = \sum u_n$ is differentiable and the derivatives are equal to the termwise differentiated series. For any $x \in (0, 3)$ and $t \geq a$, we then find that

$$u_t - \frac{1}{3} u_{xx} = \sum (u_n)_t - \frac{1}{3} \sum (u_n)_{xx} = \sum \Big(\underbrace{(u_n)_t - \frac{1}{3} (u_n)_{xx}}_{=0} \Big) = 0.$$

Since $a > 0$ is arbitrary, we have that the equation holds for all $t > 0$.