NTNU

Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4120 Mathematics 4K**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: (Code C): Approved simple calculator.

Other information:

Every answer must be justified; describe clearly how you have reached your answers.

The exam has 9 problems, 1, 2, 3, 4, 5, 6, 7, 8, 9, all of which will be given equal weight when the grade is computed.

Language: English

Number of pages: 3

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Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Students will find the examination results in Studentweb. Please contact the department if you have questions about your results. The Examinations Office will not be able to answer this.

Problem 1 (Answer in Inspera)

Let $f(x) = \begin{cases} 0, & x \in [-\pi, 0), \\ 2\pi, & x \in [0, \pi], \end{cases}$ be a function defined on $[-\pi, \pi]$.

(i) Compute the Fourier series of f. Write a number in Inspera that corresponds to the alternative you think is correct:

(1) $\pi - i \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n} e^{inx}$,

(2) $\pi - i \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{2\pi n} e^{in\pi x}$,

(3) $\pi - i \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1-(-1)^n}{n} e^{inx}$,

(4) $\pi - i \sum_{n\in\mathbb{Z}, n\neq 0} \frac{(-1)^n}{2\pi n} e^{in\pi x}$,

(5) $\pi - i \sum_{n\in\mathbb{Z}, n\neq 0} \frac{(-1)^n}{n} e^{inx}$,

(6) $\pi - i \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1-(-1)^n}{2\pi n} e^{in\pi x}$.

(ii) Let $S_f(x)$ be the sum of the Fourier series of $f(x)$. Determine the following function values and write the answers with 2 decimal precision in Inspera:

(a) $S_f(0)$ and (b) $S_f(5)$.

Problem 2 (Answer in Inspera)

Compute the the line integrals $\int_C f(z)\,dz$ in the cases

(i) $f(z) = \mathrm{Im}(z^2)$ and C is the straight line segment from $z_1 = 1$ to $z_2 = 1+3i$,

(ii) $f(z) = z + 3\cos(z)$ and C is a half circle from $z_1 = 0$ to $z_2 = i$.

Write the real and imaginary parts in Inspera with 2 decimal precision.

Problem 3 (Answer in Inspera)

The Laurent series $\dfrac{1}{(z^2-1)(z-i)} = \sum_{n=-\infty}^{\infty} a_n (z-1)^n$ converges at the point $z = \frac{5}{2}$.

Determine the largest annulus $D : r < |z - z_0| < R$ where the series converges.

Write the values for $\mathrm{Re}(z_0)$, r og R in Inspera with 2 decimal precision. Write $R = 1000$ if you find that $R = \infty$.

Problem 4 (Answer on paper)

Find a conjugate harmonic function to $u(x, y) = e^{-2y-1}\cos(2x+1) - 3x$.

Problem 5 (Answer on paper)

Find the solution $y(t)$ of the problem

$$y' + 4y + 13\int_0^t y(\tau)\,d\tau = \delta(t-4), \qquad y(0) = 0,$$

where δ is the Dirac delta-function.

Problem 6 (Answer on paper)

Show that the Fourier transform of $g(x) = \begin{cases} 0, & x < 0, \\ e^{-x}, & x \geq 0, \end{cases}$ is $\hat{g}(w) = \frac{1}{\sqrt{2\pi}} \dfrac{1}{1 + iw}$.

Then show that $u(x) = \displaystyle\int_{-\infty}^{\infty} g(x-y) f(y)\, dy$ solves the equation

$$u'(x) + u(x) = f(x) \qquad \text{for} \qquad x \in \mathbb{R}.$$

Hint: Fourier transform the equation.

Problem 7 (Answer on paper)

Determine the value of the integral

$$\int_{-\infty}^{\infty} \frac{5 - x}{(1 + x^2)(4 + x^2)}\, dx.$$

Problem 8 (Answer on paper)

Let $n \in \mathbb{N}$. Find a function $G_n(y)$ such that $u_n(x, y) = \sin(\frac{n\pi x}{2}) G_n(y)$ solves

$$(7) \qquad \begin{cases} u_{xx} + \frac{1}{4} u_{yy} = 0 & \text{for} \quad x \in (0, 2),\ y \in (0, 3), \\ u(0, y) = 0 = u(2, y) & \text{for} \quad y \in [0, 3], \\ u(x, 0) = 0 & \text{for} \quad x \in [0, 2]. \end{cases}$$

Then find the solution u of problem (7) and the boundary condition

$$u(x, 3) = 11 \sin(\pi x) + \sin(5\pi x).$$

Problem 9 (Answer on paper)

Show that $u(x, t) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{1}{3} n^2 t} \sin(nx)$ converges uniformly and solves the equation

$$u_t - \tfrac{1}{3} u_{xx} = 0, \qquad x \in (0, \pi),\ t > 0.$$

Hint: Weierstrass M-test and term-wise differentiation.

Miscellaneous

- **Heaviside/unit step function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero except at $t = a$, $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, and $\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$ for any continuous function g.
- **Convolution**

  For functions defined on the real line:
  $$f * g\,(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}.$$
  For functions defined only on the positive half-axis:
  $$f * g\,(x) = \int_0^x f(y)g(x-y)dy, \quad x > 0.$$

Laplace transform

- Definition: $\mathcal{L}[f](s) = F(s) = \int_0^\infty f(t)e^{-st}dt$

General formulas		$f(t)$	$F(s)$
		1	$\frac{1}{s}$
$\mathcal{L}[e^{at}f(t)](s) = F(s-a)$		$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$		e^{at}	$\frac{1}{s-a}$
$\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$		$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right](s) = \frac{1}{s}\mathcal{L}[f](s)$		$\cos bt$	$\frac{s}{s^2+b^2}$
$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$		$\sin bt$	$\frac{b}{s^2+b^2}$
$\mathcal{L}[f(t-c)u(t-c)](s) = e^{-cs}F(s),\ c > 0$		$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{L}[tf(t)](s) = -F'(s)$		$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty F(\sigma)d\sigma$		$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
		$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- $2L$-periodic functions, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}},$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx,$$

$$c_0 = a_0, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx, \qquad c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \bar{c}_n.$$

- Functions defined on the whole real line (need not be periodic)

$$\hat{f}(w) = \mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx,$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw.$$

- Parseval's identities

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2, \qquad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw$$

General formulas		$f(x)$	$\hat{f}(w)$
$\widehat{f'(x)} = iw\hat{f}(w)$		$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-iaw}$
$\widehat{f''(x)} = -w^2\hat{f}(w)$		$\begin{cases} 1, & -b \le x \le b \\ 0, & \lvert x\rvert > b \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
$f\widehat{(x-a)} = e^{-iaw}\hat{f}(w)$		$e^{-ax}u(x)$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\hat{f}(w-b) = e^{\widehat{ibx}f(x)}$		$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$
$\widehat{f*g} = \sqrt{2\pi}\hat{f}\hat{g}$		e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i \sin y)$
- $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}, \qquad \cosh z = \frac{e^{z}+e^{-z}}{2}, \quad \sinh z = \frac{e^{z}-e^{-z}}{2}$
- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad b_n = \frac{1}{2\pi i} \oint_C f(z)(z-z_0)^{n-1} dz$$

Some useful integrals

$\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$

$\int x \cos ax\, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$

$\int x^2 \sin ax\, dx = \frac{2}{a^2} x \sin ax + \frac{2-a^2x^2}{a^3} \cos ax + C$

$\int x^2 \cos ax\, dx = \frac{2}{a^2} x \cos ax - \frac{2-a^2x^2}{a^3} \sin ax + C$

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$

$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \ a > 0$

Some trigonometric identities

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

Some important series

- $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, $\displaystyle\sum_{n=0}^{\infty} x^n$ diverges for $|x| \geq 1$.
- $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for $x \in \mathbb{R}$.

Linear second order differential equations

Let r_1 and r_2 solve $r^2 + ar + b = 0$. Then

$$y''(x) + ay'(x) + by = 0$$

has general solution given by:

$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ if $r_1 \neq r_2, \quad r_1, r_2 \in \mathbb{R}$,

$y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$ if $r_1 = r_2, \quad r_1, r_2 \in \mathbb{R}$,

$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ if $r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta, \quad \alpha, \beta \in \mathbb{R}$.