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EXAM IN MATHEMATICS 4N (TMA4130)

Saturday December 8 2007
Time: 09:00 – 13:00 Final grades: Januar 7 2008

Permitted Aids:
Approved calculator.
Formula sheet (handed out with the exam).

Problem 1 Find the function $y(t)$, $t \geq 0$ such that

$$\int_0^t y'(u)y(t-u)dt = t^2,\ t > 0,$$

and $y(0) = 0$.

Problem 2

a) Given the function on $(0, 2\pi)$

$$f(x) = \begin{cases} x, & \text{if } 0 < x < \pi; \\ 2\pi - x, & \text{if } \pi < x < 2\pi. \end{cases}$$

find the sin-Fourier series for f.

b) Find all solutions having the form $u(x,t) = X(x)T(t)$ of the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u, \ 0 < x < 2\pi, t > 0$$
$$u(0,t) = 0, \quad u(2\pi, t) = 0$$

c) Find a solution to the problem in part **b)** such that

$$u(x,0) = f(x), \ 0 < x < 2\pi,$$

where the function f is defined in part **a)**

Problem 3 Let a function f on $(-\infty, \infty)$ is defined as

$$f(x) = \begin{cases} \cos x, & \text{if } |x| < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Find the Fourier transform of f and then evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin 2w}{w} \cos w \, dw.$$

Problem 4 You are given the problem

$$\begin{aligned} u''' + (3 - u')u'' + 2u &= 0 \\ u(0) &= 1 \\ u'(0) &= 2 \\ u''(0) &= 5. \end{aligned} \tag{1}$$

a) Write the problem as a system of equations.

Heun's method can be viewed as a predictor-corrector combination of Euler's method and the trapezoidal rule.

Backward Euler is given by

$$\underline{u}_{n+1} = \underline{u}_n + hf\left(t_{n+1}, \underline{u}_{n+1}\right).$$

Give a method using Euler's method as a predictor and backward Euler as a corrector.

b) Apply one step of the method you obtained in **a)** to (??). Use $h = 0.1$.

If you did not manage to find the method in **a)**, use Heun's method instead.

Problem 5

a) Find the polynomial $p_2(x)$ which interpolates

x_k	-2	-1	2
f_k	-13	-5	7

using Lagrangian interpolation.

b) We then add another datapoint. We now want to find the polynomial $p_3(x)$ of the lowest possible degree which interpolates the data set

t_k	-2	-1	2	3
f_k	-13	-5	7	4

.

You can choose how you find this polynomial yourself.

Numerics formulae

- Error in polynomial interpolation

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n}(x - x_i).$$

If the nodes are equidistant (including the end points) and $\left|f^{(n+1)}(\xi)\right| \leq M$ this yields

$$|f(x) - p(x)| \leq \frac{1}{4(n+1)} M \left(\frac{b-a}{n}\right)^{n+1}.$$

- Numerical differentiation

$$\begin{aligned}
f'(x) &= \frac{1}{h}\big(f(x+h) - f(x)\big) + \frac{h}{2} f''(\xi) \\
f'(x) &= \frac{1}{2h}\big(f(x+h) - f(x-h)\big) - \frac{h^2}{6} f'''(\xi) \\
f''(x) &= \frac{1}{h^2}\big(f(x+h) - 2f(x) + f(x-h)\big) - \frac{h^2}{12} f^{(4)}(\xi).
\end{aligned}$$

- Newton's method for the resolution of systems of non-linear equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$

$$\mathbf{J}^{(k)} \cdot \mathbf{\Delta x}^{(k)} = -\mathbf{f}(\mathbf{x}^{(k)}),$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{\Delta x}^{(k)}.$$

- Iterative methods for systems of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \cdots, n$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \big(b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k)}\big) \qquad \text{(Jacobi)}$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \big(b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k)}\big). \qquad \text{(Gauss–Seidel)}$$

- A second order Runge-Kutta method (Heun) for initial value problems

$$\begin{aligned}
\mathbf{k}_1 &= h\mathbf{f}\left(x_n, \mathbf{y}_n\right) \\
\mathbf{k}_2 &= h\mathbf{f}\left(x_n + h, \mathbf{y}_n + h\mathbf{k}_1\right) \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{1}{2}\left(\mathbf{k}_1 + \mathbf{k}_2\right)
\end{aligned}$$

Also note that there are a few formulae relating to numerics in "Appendix A" of *K. Rottmann: Matematisk formelsamling*.

Table of Laplace transforms

$f(t)$	$\mathcal{L}(f)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^n \ (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$