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Academic responsibility:
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FINAL EXAMINATION IN CALCULUS 4M (TMA4122)

Wednesday the 13th of December 2006
Time: 09:00 – 13:00 Grading due the 12th of January 2007

Examination support:

- K. Rottmann: *Matematisk Formelsamling*
- Simple pocket calculator (HP 30S).

General information:

- All subproblems count equally towards the examination results.
- All answers must be justified.
- Any answer must be accompanied by sufficient calculations to allow assessement of which methods and intermediate results are being used.
- If two functions $f_1(x)$ and $f_2(x)$ are sufficiently differentiable, then

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(f_1(x)\,f_2(x)\big) = \sum_{k=0}^{n} \binom{n}{k} f_1^{(k)}(x)\,f_2^{(n-k)}(x) \quad \text{in which} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

- A collection of additional formulae in numerics is available at the back of the problem set.
- Good luck!

Problem 1 Suppose a and b are real constants and let (1) and (2) be the boundary value problems

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{when } t > 0 \text{ and } x \in (0,1), \\ u(0,t) = a, & \text{when } t > 0, \\ u(1,t) = b, & \text{when } t > 0, \end{cases} \tag{1}$$

and

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial^2 v}{\partial x^2} = 0, & \text{when } t > 0 \text{ and } x \in (0,1), \\ v(0,t) = 0, & \text{when } t > 0, \\ v(1,t) = 0, & \text{when } t > 0. \end{cases} \tag{2}$$

a) If the functions u_1 and u_2 both satisfy the boundary value problem (1) (i.e., if both u_1 and u_2 resolve (1)), what boundary value problems are satisfied by the functions $u_1 + u_2$ and $u_1 - u_2$ respectively? Does the principle of superposition hold for the *boundary value problems* (1) and (2)?

b) Suppose $u(x,t)$ is a solution to the boundary value problem (1) and define the function $v(x,t)$ by

$$v(x,t) = u(x,t) - \big(a + (b-a)x\big) \quad \text{when } t \geq 0 \text{ and } x \in [0,1].$$

Prove that $v(x,t)$ then resolves the boundary value problem (2).

Find all solutions to (2) of the form

$$v(x,t) = F(x)G(t).$$

c) Let $a = -1$ and $b = 1$ in (1). Find the solution $u(x,t)$ of the initial/boundary value problem given by (1) and the initial value

$$u(x,0) = \sin(\pi x), \quad \text{when } 0 < x < 1. \tag{3}$$

Problem 2 The function

$$f(x) = \frac{e^{-x}}{1+x}$$

is defined for all $x \geq 0$.

a) Compute $S(x)$, the natural, cubic spline interpolating $f(x)$ at the points 0, 2, 6, and 8. Also, determine the error, $S(x) - f(x)$, at the point $x = 1.25$.

b) Compute an approximation to the integral $\int_0^8 f(x)\,\mathrm{d}x$ using Simpson's method with 8 equally sized sub-intervals.

c) How many equally sized intervals will minimally be needed to guarantee that the result of Simpson's method in **b)** is correct to 6 decimals—that is, such that the absolute value of the error is less than $\frac{1}{2} \cdot 10^{-6}$.

Problem 3 The system of non-linear equations

$$\begin{aligned} \sin(x_1) + \mathrm{e}^{x_2} - 3 &= 0 \\ (x_2 + 3)^2 - x_1 - 4 &= 0 \end{aligned} \tag{4}$$

has a solution near the point $(12, 1)$.

Perform a single iteration of Newton's method to compute a better approximation to the root $\mathbf{r}$ of the system (4).

Problem 4 In an anisotropic material where the heat conductivity is twice as high in the vertical direction as in the horizontal direction, the steady heat equation takes the form

$$u_{xx} + 2u_{yy} = 0. \tag{5}$$

We wish to numerically resolve (5) in a square of side length 1 using the boundary conditions

$$\begin{aligned} u(x, 0) = u(0, y) &= 0 \\ u(x, 1) = u(1, y) &= 1 \end{aligned}$$

for x, y in $[0, 1]$. We consider the grid in Figure 1 in which $h = \frac{1}{3}$ and $U_{i,j} \approx u(ih, jh)$.

a) Show that the difference scheme for the steady heat equation (5) is given by

$$U_{i+1,j} + U_{i-1,j} + 2U_{i,j+1} + 2U_{i,j-1} - 6U_{i,j} = 0$$

for all i and j.

b) Establish a linear system of equations satisfied by the quantities $U_{i,j}$ when $i, j = 1, 2$ and perform a single Gauss–Seidel iteration on this system, starting from the initial point

$$U_{1,1}^{(0)} = U_{2,1}^{(0)} = U_{1,2}^{(0)} = U_{2,2}^{(0)} = \frac{1}{2}.$$

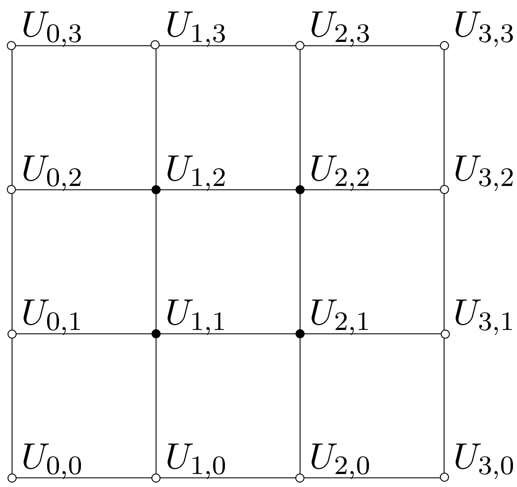

Figure 1: Grid for the difference scheme in Problem 4.

Happy holidays!