Fourier series and applications

1. – Write down the Fourier series for 2π-periodic function f defined by $f(t) = t\sin 2t$ for $0 < t < 2\pi$

   – Write down the Fourier series for odd 2π-periodic function f defined by $f(t) = t\sin 2t$ for $0 < t < \pi$

   – Write down the Fourier series for even 2π-periodic function f defined by $f(t) = t\sin 2t$ for $0 < t < \pi$

2. Write down the Fourier series for 2π-periodic functions $f_1(t) = \cos^3 t$, $f_2(t) = \cos 3t$, and $f_3(t) = (\cos 2t + \sin 2t)^2$.

3. Let 2π-periodic functions f_1, f_2, and f_3 be defined by the relations

$$f_1(t) = \begin{cases} t, & 0 < t < \pi, \\ -t, & -\pi < t < 0. \end{cases} \quad f_2(t) = \begin{cases} \pi - t, & 0 < t < \pi, \\ \pi + t, & -\pi < t < 0. \end{cases} \quad f_1(t) = \begin{cases} t, & -\pi/2 < t < \pi/2, \\ \pi - t, & \pi/2 < t < 3\pi/2. \end{cases}$$

   – Sketch the graphs of these functions and find their Fourier series;

   – Can you obtain the Fourier series for f_2 and f_3 directly from the Fourier series for f_1 ?

4. Find all cos-coefficients of the 2π-periodic function defined by $f(t) = t^3$ for $-\pi < t < \pi$.

5. Find sin- and cos- half-range expansion for the functions defined by the relations $f_1(t) = t^3$ for $0 < t < \pi$, $f_2(t) = \cos t$ for $0 < t < \pi$.

6. Write down the general formula for Fourier series and their coefficients for functions with period 2.

7. Let f be a function of period 2 with $f(t) = t^2$ if $0 < t < 2$.

   – plot the graph of f;

   – find the Fourier series for f;

   – find the sums of this series at the points $t = 0$ and at $t = 1$;

   – prove the following relations:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6},$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12},$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.$$

8. – Write down the general formula for complex Fourier series and their coefficients for functions with period 2.

   – Find the complex Fourier series for the function $f(t) = e^t$ on the interval $[-1, 1]$

9. Find the temperature $u(x,t)$ at any time in a metal rod 50 cm long, insulated on the sides, which initially has a uniform temperature $20^o C$ throughout and whose ends are maintained at $0^o C$ for all $t > 0$.

10. (*a bit more complicated*) Solve the heat conduction problem:

$$u_{xx} = u_t, , \quad 0 < x < 30, \quad t > 0,$$

$$u(0,t) = 0, \quad u(30,t) = 20, \quad t > 0,$$

$$u(x,0) = 60 - 2x, \quad 0 < x < 30.$$

11. Consider a vibrating string of length $L = 30$ that satisfies the wave equation

$$4u_{xx} = u_{tt}, \quad 0 < x < 30, \quad t > 0.$$

   Assume that the ends of string are fixed and that the string is set in motion with no initial velocity from the initial position

$$u(x,0) = f(x) = \begin{cases} x/10, & 0 < x < 10 \\ (30 - x/20), & 10 < x < 30. \end{cases}$$

   Find the displacement $u(x,t)$ of the string and describe its motion.