Solutions, mathematical part

1a. Find the inverse Laplace transform of the function

$$F(s) = \frac{1}{s(s^2+3s+2)}e^{-2s}$$

We have

$$\frac{1}{s(s^2+3s+2)}e^{-2s} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)},$$

so the shift theorem gives

$$\mathcal{L}^{-1}F(t) = \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]u(t-2) = \frac{1}{2}\left[1 - 2e^2e^{-t} + e^-4e^{-2t}\right]u(t-2).$$

1b. Solve the initial value problem

$$y'' + 3y' + 2y = f(t), \ \ y(0) = 0, \ y(1) = 1,$$

$$f(t) = \begin{cases} 1, & 0 < t < 1; \\ 0, & t > 1. \end{cases}$$

We have $f(t) = u(t) - u(t-1)$, so

$$\mathcal{L}f = \frac{1}{s}(1 - e^{-s}).$$

Applying the Laplace transform and using that $y(0) = 0$ we obtain

$$(s^2+3s+2)Y(s) - y'(0) = \frac{1}{s}(1-e^{-s}),$$

hence

$$Y(s) = \frac{y'(0)}{s^2+3s+2} + \frac{1}{s(s^2+3s+2)} - \frac{1}{s(s^2+3s+2)}e^{-s}.$$

Using the expansion

$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

as well as expansions from the part **1a** we obtain

$$y(t) = y'(0)(e^{-t} - e^{-2t}) + \left(\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right) - \left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right) u(t-1).$$

Now the condition $y(1) = 1$ yields

$$y'(0) = \frac{e^2 + 2e - 1}{2(1+e)},$$

so the final solution has the form

$$y(t) = \frac{e^2 + 2e - 1}{2(1+e)}(e^{-t} - e^{-2t}) + \left(\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right) - \left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right) u(t-1).$$

Remark. For those students who considered this too complicated I offered a simplified version with replacing the condition $y(1) = 1$ by $y'(0) = 1$. They should obtain

$$y(t) = (e^{-t} - e^{-2t}) + \left(\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right) - \left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right) u(t-1).$$

If doing this correct they will also obtain full credit.

1c. Solve the integral equation

$$y(t) = \cos t + \int_0^t y(\tau)\cos(\tau - t)d\tau, \ t > 0.$$

First we use that cos is an even function and rewrite the equation as

$$y(t) = \cos t + \int_0^t y(\tau)\cos(t - \tau)d\tau, \ t > 0,$$

so it is now clear that we have convolution. Since

$$\mathcal{L}(\cos t) = \frac{s}{1+s^2},$$

we obtain

$$Y(s) = \frac{s}{1+s^2} + \frac{s}{1+s^2}Y(s).$$

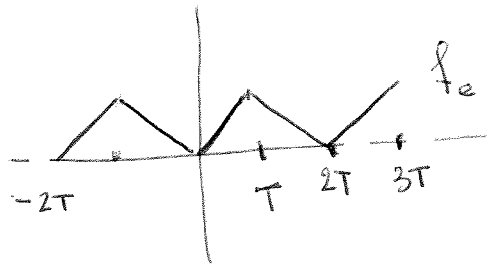
Therefore

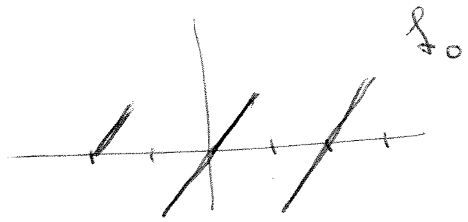
$$Y(s) = \frac{s}{1-s+s^2} = \frac{s-1/2}{(s-1/2)^2+3/4} + \frac{1}{2}\frac{1}{(s-1/2)^2+3/4}.$$

Finally

$$y(t) = e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right).$$

2a. Let $f(t) = t$ for $0 < t < T$. Sketch the even (f_e) and odd (f_o) prolongations of f on the segment $[-2T, 3T]$. Find the Fourier series for f_e.





Evaluate the Fourier series of f_e:

$$a_0 = \frac{1}{T}\int_0^T t\,dt = \frac{1}{2}T;$$

$$a_n = \frac{2}{T}\int_0^T t\cos\frac{n\pi}{T}t\,dt == \begin{cases} -\frac{4T}{(n\pi)^2}, & n \text{ is odd;} \\ 0, & n \text{ is even.} \end{cases}$$

Finally:

$$f_e(t) \asymp \frac{T}{2} - \frac{4T}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\cos\frac{(2l+1)\pi}{T}t.$$

2b. Find the sum

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}\cdots.$$

We have

$$0 = f_e(0) = \frac{T}{2} - \frac{4T}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}.$$

Therefore

$$\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2} = \frac{\pi^2}{8}.$$

2c. Consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 \le x \le 3, \ t > 0;$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \ \frac{\partial u}{\partial x}(3,t) = 0, \ \ t > 0$$

Find all solution to this problem having the form $u(x,t) = F(x)G(t)$.

Let $u(x,t) = F(x)G(t)$. Then the problem can be reduced to two problems for ordinary differential equations

$$\frac{\dot{G}}{G} = k, \ \text{for } t > 0,$$

and

$$\frac{F''}{F} = k, \ \text{for } 0 < x < 3, \qquad F'(0) = F'(3) = 0,$$

here k is a constant to be found.

A simple analysis (you have to explain how it is carried out) shows that the second equation can be consistent only if

$$k = k_n = -(\frac{n\pi}{3})^2, \quad n = 0, 1, 2, \dots .$$

In the case $n = 0$ we have $k_0 = 0$ and $F = A_0$, $G = B_0$, just some constants. For $n > 0$ the solutions are

$$F_n(x) = A_n \cos\frac{n\pi x}{3}; \quad G_n(t) = B_n e^{-t(\frac{n\pi}{3})^2}.$$

When taking the product FG we do not need two arbitrary constants. So the general solution has the form

$$u_n(x,t) = A_n e^{-t(\frac{n\pi}{3})^2} \cos\frac{n\pi x}{3}, \qquad n = 0, 1, 2, \dots .$$

2d. Find solutions to the problem from **2c** satisfying the initial conditions

- (i) $u(x,0) = (\cos\frac{\pi}{3}x + \sin\frac{\pi}{3}x)^2, \ 0 \le x \le 3$
- (ii) $u(x,0) = x, \ 0 \le x \le 3.$

We look for the solution in the form

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-t(\frac{n\pi}{3})^2} \cos\frac{n\pi x}{3},$$

and the coefficients A_n are to be chosen in order to meet the initial conditions.

That is why

$$A_n = \frac{2}{3}\int_0^3 u(x,0)\cos\frac{n\pi}{3}x dx, \ n > 0,$$

and

$$A_0 = \frac{1}{3}\int_0^3 u(x,0)dx.$$

So it remains to substitute the values of $u(x,0)$ from the conditions **(i)** and **(ii)** and then evaluate the integral.

(i). We have

$$(\cos\frac{\pi}{3}x + \sin\frac{\pi}{3}x)^2 = 1 + \sin\frac{2\pi}{3}x$$

and simple calculation shows $A_0 = 1$. For $n > 0$ we have

$$A_n = \frac{2}{3}\int_0^3 \sin\frac{2\pi}{3}x\cos\frac{n\pi}{3}x dx.$$

We integrate by parts:

$$\frac{2}{3}\int_0^3 \sin\frac{2\pi}{3}x\cos\frac{n\pi}{3}x dx = -\frac{1}{\pi}((-1)^n - 1) - \frac{n}{3}\int_0^3 \cos\frac{2\pi}{3}x\sin\frac{n\pi}{3}x dx$$

The integral in the righthand side integrate by parts once again:

$$\frac{2}{3}\int_0^3 \sin\frac{2\pi}{3}x\cos\frac{n\pi}{3}x dx = -\frac{1}{\pi}((-1)^n - 1) + \frac{n^2}{4}\frac{2}{3}\int_0^3 \sin\frac{2\pi}{3}x\cos\frac{n\pi}{3}x dx,$$

or

$$\left(\frac{n^2}{4} - 1\right)A_n == \begin{cases} -\frac{2}{\pi}, & n \ is \text{ odd;} \\ 0, & n \ is \text{ even} \end{cases}$$

Also an explicit calculation shows $A_2 = 0$. Finally we obtain the expansion

$$u(x,t) = 1 - \frac{8}{\pi}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2 - 4}e^{-t(\frac{(2l+1)\pi}{3})^2}\cos\frac{(2l+1)\pi x}{3},$$

Remark. You could also evaluate the integral if representing

$$\sin\frac{2\pi}{3}x = \frac{1}{2i}(e^{i\frac{2\pi}{3}x} - e^{-i\frac{2\pi}{3}x}), \quad \cos\frac{n\pi}{3}x = \frac{1}{2}(e^{i\frac{n\pi}{3}x} + e^{-i\frac{n\pi}{3}x}),$$

then the integral can be calculated directly.

(ii). We use the results of Problem **2a** with $T = 3$. We then have

$$x = u(x,0) = \frac{3}{2} - \frac{3T}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\cos\frac{(2l+1)\pi}{3}x,$$

and

$$u(x,t) = \frac{3}{2} - \frac{3T}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}e^{-t(\frac{(2l+1)\pi}{3})^2}\cos\frac{(2l+1)\pi}{3}x,$$

3. Find the Fourier transform of the function $g(t) = e^{-2|t|}$ and represent g as an inverse Fourier transform.

We have

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{2t-itw}dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-2t-itw}dt =$$

$$\frac{1}{\sqrt{2\pi}}\frac{1}{2-iw} + \frac{1}{\sqrt{2\pi}}\frac{1}{2+iw} = \frac{1}{\sqrt{2\pi}}\frac{4}{4+w^2}.$$

Respectively

$$g(t) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{4+w^2}e^{itw}dw$$

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Oppgave 4

a) Det sies ikke noe om hvilken metode som skal brukes i oppgaven, så studenten kan velge fritt, eller gjøre begge.

Med Newtoninterpolasjon:

x	$f(x)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
0	$\boxed{-1}$		
		$\boxed{0}$	
1	-1		$\boxed{1}$
		2	
2	-1		

Vi får da

$$\begin{aligned} p_2(x) &= -1 + 0(x-0) + 1(x-0)(x-1) \\ &= \underline{\underline{x^2 - x - 1}} \end{aligned}$$

Med Lagrangeinterpolasjon: Vi har først kardinalfunksjonene

$$\begin{aligned} \ell_1(x) &= \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x^2 - 3x + 2) \\ \ell_2(x) &= \frac{(x-0)(x-2)}{(1-0)(1-2)} = -(x^2 - 2x) \\ \ell_3(x) &= \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}(x^2 - x) \end{aligned}$$

og vi får

$$\begin{aligned} p_2(x) &= (-1)\frac{1}{2}(x^2 - 3x + 2) - (-1)(x^2 - 2x) + 1\frac{1}{2}(x^2 - x) \\ &= \underline{\underline{x^2 - x - 1}} \end{aligned}$$

b) Med trapesmetoden får vi:

$$\begin{aligned} \int_0^2 f(x)\,\mathrm{d}x &\approx \frac{1}{2}f(0) + f(1) + \frac{1}{2}f(2) \\ &= \frac{1}{2}(-1) + (-1) + \frac{1}{2}1 \\ &= \underline{\underline{-1}} \end{aligned}$$

Med Simpsons metode får vi:

$$\begin{aligned} \int_0^2 f(x)\,\mathrm{d}x &\approx \frac{1}{3}\left(f(0) + 4f(1) + f(2)\right) \\ &= \frac{1}{3}(-1 + 4 - 1) = \underline{\underline{-\frac{4}{3}}} \end{aligned}$$

Vi beregner så intergralet av vårt interpolerende polynom eksakt,

$$\begin{aligned} \int_0^2 p_2(x)\,\mathrm{d}x &= \int_0^2 (x^2 - x - 1)\,\mathrm{d}x \\ &= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - x\right]_0^2 \\ &= \frac{1}{3}8 - \frac{1}{2}4 - 2 = \underline{\underline{-\frac{4}{3}}} \end{aligned}$$

Eksakt integrasjon av $p_2(x)$ er den beste approksimasjonen vi kan få til $\int_0^2 f(x)\,\mathrm{d}x$. Simpsons metode gir samme resultatet, fordi Simpsons metode integrerer alle polynomer av grad 3 eller lavere eksakt, og utnytter dermed måledataene fra f så godt som det er mulig. Trapesmetoden integrerer bare polynom av grad 1 eksakt, og gir derfor ikke like bra svar.

c)

$$p_2(4) = \underline{\underline{11}}$$

For å estimere feilen under ekstrapolasjon, gjelder fortsatt den første formelen for feil i interpolasjon som gitt på formelarket, bortsett fra at man må utvide intervallet hvor ξ i $f^{(n+1)}(\xi)$ kan være i. Her er gitt at $|f^{(3)}(x)| < 1$ for $0 \leq x \leq 4$, og vi får da

$$|f(4) - p_2(4)| < \frac{1}{3!}1(4-0)(4-1)(4-2) = \underline{\underline{4}}$$

Oppgave 5 Vi løser diffusjonsligningen $u_t = u_{xx}$, og skal finne $u_{1,1}$ og $u_{2,1}$ i gitteret

$$\begin{array}{cccc} \bullet\, 1 & \bullet\, u_{1,1} & \bullet\, u_{2,1} & \bullet\, 1 \\ \bullet\, 1 & \bullet\, \frac{1}{4} & \bullet\, \frac{1}{4} & \bullet\, 1 \end{array}$$

hvor randverdier er puttet inn i gitteret. Vi bruker Couranttallet

$$r = \frac{\Delta t}{(\Delta x)^2} = \frac{0.2}{(1/3)^2} = 1.8$$

i beregningene under.

a) Forlengs Euler i tid og sentraldifferanse i rom gir formelen

$$u_{i,j+1} = (1-2r)u_{i,j} + r(u_{i+1,j} + u_{i-1,j})$$

som for $i = 1, 2$ og $j = 0$ gir tallsvarene

$$u_{1,1} = (1 - 2 \cdot 1.8)\frac{1}{4} + 1.8(\frac{1}{4} + 1) = \underline{\underline{1.6}}$$
$$u_{2,1} = (1 - 2 \cdot 1.8)\frac{1}{4} + 1.8(1 + \frac{1}{4}) = \underline{\underline{1.6}}$$

b) Crank–Nicolsons metode er gitt ved formelen

$$(2+2r)u_{i,j+1} - r(u_{i+1,j+1} + u_{i-1,j+1}) = (2-2r)u_{i,j} + r(u_{i+1,j} + u_{i-1,j})$$

for $i = 1, 2$ og $j = 0$ får vi de to ligningene

$$5.6u_{1,1} - 1.8(1 + u_{2,1}) = -1.6\frac{1}{4} + 1.8(1 + \frac{1}{4})$$
$$5.6u_{2,1} - 1.8(u_{1,1} + 1) = -1.6\frac{1}{4} + 1.8(\frac{1}{4} + 1)$$

som gir svaret $u_{1,1} = u_{2,1} \approx \underline{\underline{0.961}}$.

Man kan også fra starten av oppgaven observere at symmetrien i problemet gir at $u_{1,1} = u_{2,1}$ og man kan da slippe unna med mindre regning.

Fysikken i problemet gir at u aldri kan bli større enn 1, siden diffusjon vil sørge for å utjevne alle forskjeller, og konsentrasjonen langs kanten holdes konstant til 1. Svaret fra **a)** må derfor være galt. Fra stabilitetsanalyse vet vi at metoden er nødt til å ha $r = \frac{\Delta t}{(\Delta x)^2} \leq 0.5$ for stabil løsning, og det har vi ikke her, så vi kan ikke vente annet fra metoden enn at tallsvarene kan bli riv ruskende gale.

Crank–Nicolson er stabil for alle Courant-tall r, og svaret den gir i dette tilfellet kan vi derfor ha tiltro til.