TMA4125 Matematikk

4N

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Solutions to exercise set 3

$\boxed{5.6.13}$ Derive the following formula, showing the details of your work:

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \cosh at \cos at$$

Expand:

$$\frac{s^3}{s^4+4a^4} = \frac{s^3}{s^4+4a^4+(4a^2s^2-4a^2s^2)}$$

$$\frac{s^3}{(s^4+4a^2s^2+4a^4)-4a^2s^2} = \frac{s^3}{(s^2+2a^2)^2-(2as)^2}$$

Since $x^2-y^2=(x+y)(x-y)$:

$$\frac{s^3}{(s^2+2a^2)^2-(2as)^2} = \frac{s^3}{(s^2+2sa+2a^2)(s^2-2sa+2a^2)}$$

Partial fractions with complex, non-repeated factors:

$$\frac{s^3}{(s^2+2sa+2a^2)(s^2-2sa+2a^2)} = \frac{As+B}{s^2+2sa+2a^2} + \frac{Ms+N}{s^2-2sa+2a^2}$$

$$\begin{aligned} A &= 1/2 \\ B &= a/2 \\ M &= 1/2 \\ N &= -a/2 \end{aligned}$$

Then we have

$$\frac{1}{2}\frac{s+a}{s^2+2sa+2a^2} + \frac{1}{2}\frac{s-a}{s^2-2sa+2a^2} = \frac{1}{2}\frac{s+a}{(s+a)^2+a^2} + \frac{1}{2}\frac{s-a}{(s-a)^2+a^2}$$

Use the first shifting theorem:

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s+a}{(s+a)^2+a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s-a}{(s-a)^2+a^2}\right\} = e^{-at}\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s}{s^2+a^2}\right\} + e^{at}\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s}{s^2+a^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s+a}{(s+a)^2+a^2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s-a}{(s-a)^2+a^2}\right\}=e^{-at}\frac{\cos at}{2}+e^{at}\frac{\cos at}{2}$$

$$e^{-at}\frac{\cos at}{2}+e^{at}\frac{\cos at}{2}=\cos at\left(\frac{e^{-at}+e^{at}}{2}\right)=\cosh at\cos at$$