Excercise

Find the function $y(t)$, $t > 0$ satisfying the differential equation

$$y''(t) - 2y'(t) + y(t) = 1,\ t > 0$$

and such that

$$y(0) = 0,\ y(1) = 10.$$

Solution

$$y''(t) - 2y'(t) + y(t) = 1$$

Do the Laplace Transform and define $y'(0) = y_1$:

$$s^2Y - sy(0) - y_1 - 2sY + 2y(0) + Y = 1/s$$

Solve with regards to Y:

$$Y = \frac{1/s + y_1}{s^2 - 2s + 1} = \frac{1}{s}\frac{1}{(s-1)^2} + \frac{y_1}{(s-1)^2}$$

Find the inverse:

$$y(t) = \int_0^t \tau e^\tau \mathrm{d}\tau + y_1 te^t = te^t - e^t + 1 + y_1 te^t = (y_1 + 1)te^t - e^t + 1$$

Use the last initial conditions to find y_1:

$$y(1) = (y_1 + 1)e - e + 1 = 10$$

$$y_1 = \frac{9}{e}$$

Then we have

$$y(t) = \left(\frac{9}{e} + 1\right) te^t - e^t + 1$$