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Solutions to exercise set 4

5.7.3 We are to solve the initial value problem:

$$\begin{aligned} y_1' &= -y_1 + 4y_2 \\ y_2' &= 3y_1 - 2y_2 \\ y_1(0) &= 3 \\ y_2(0) &= 4 \end{aligned}$$

First we do the Laplace transform on the two equations:

$$\begin{aligned} sY_1 - 3 &= -Y_1 + 4Y_2 \\ sY_2 - 4 &= 3Y_1 - 2Y_2 \end{aligned}$$

This must be solved algebraically for $Y_1(s)$ and $Y_2(s)$

$$\begin{aligned} Y_1(s+1) &= 4Y_2 + 3 \\ Y_2(s+2) &= 3Y_1 + 4 \end{aligned}$$

Insert $Y_2 = \frac{3Y_1}{s+2} + \frac{4}{s+2}$ into the exspression for Y_1:

$$\begin{aligned} (s+1)Y_1 &= 3 + \frac{12Y_1}{s+2} + \frac{16}{s+2} \\ (s+2)(s+1)Y_1 &= 3(s+2) + 12Y_1 + 16 \\ (s^2+3s-10)Y_1 &= 3s + 22 \\ Y_1 &= \frac{3s+22}{s^2+3s-10} \end{aligned}$$

Then insert this into the other equation:

$$\begin{aligned} (s+2)Y_2 &= 4 + \frac{9s+66}{s^2+3s-10} \\ Y_2 &= \frac{4}{s+2} + \frac{9s+66}{(s+2)(s^2+3s-10)} \end{aligned}$$

Then we take a look at $s^2 + 3s - 10$:

$$\begin{aligned} s^2+3s-10 &= (s+\frac{3}{2})^2-(\frac{7}{2})^2 \\ &= \left(s+\frac{3}{2}+\frac{7}{2}\right)\left(s+\frac{3}{2}-\frac{7}{2}\right) \\ &= (s+5)(s-2) \end{aligned}$$

We have then:

$$\begin{aligned} Y_1 &= \frac{3s+22}{(s+5)(s-2)} \\ Y_2 &= \frac{4}{s+2}+\frac{9s+66}{(s+2)(s+5)(s-2)} \end{aligned}$$

Some rewriting:

$$\begin{aligned} Y_1 &= \frac{(s+5)3+7}{(s+5)(s-2)} = \frac{3}{s-2}+\frac{7}{(s+5)(s-2)} \\ Y_2 &= \frac{4}{s+2}+\frac{(s+5)9+21}{(s+2)(s+5)(s-2)} = \frac{4}{s+2}+\frac{9}{s^2-4}+\frac{21}{(s^2-4)(s+5)} \end{aligned}$$

Partial fractions:

$$\frac{7}{(s+5)(s-2)} = \frac{A}{s+5}+\frac{B}{s-2}$$

$$\begin{aligned} A &= -1 \\ B &= 1 \end{aligned}$$

and

$$\frac{21}{(s^2-4)(s+5)} = \frac{As+B}{s^2-4}+\frac{C}{s+5}$$

$$\begin{aligned} A &= -1 \\ B &= 5 \\ C &= 1 \end{aligned}$$

Then we have:

$$\begin{aligned} Y_1 &= \frac{3}{s-2}+\frac{-1}{s+5}+\frac{1}{s-2} = \frac{4}{s-2}-\frac{1}{s+5} \\ Y_2 &= \frac{4}{s+2}+\frac{9}{s^2-4}-\frac{s-5}{s^2-4}+\frac{1}{s+5} = \frac{4}{s+2}+\frac{14}{s^2-4}-\frac{s}{s^2-4}+\frac{1}{s+5} \end{aligned}$$

Then the inverse Laplace gives us:

$$
\begin{aligned}
y_1 &= -e^{-5t} + 4e^{2t} \\
y_2 &= 4e^{-2t} + 14\frac{\sinh 2t}{2} - \cosh 2t + e^{-5t} \\
&= 4e^{-2t} + \frac{7}{2}e^{2t} - \frac{7}{2}e^{-2t} - \frac{e^{-2t}}{2} - \quad + e^{-5t} \\
&= 3e^{2t} + e^{-5t}
\end{aligned}
$$

5.7.13 We are to solve the initial value problem:

$$
\begin{aligned}
y_1' &= 2y_1 - 4y_2 + u(t-1)e^t \\
y_2' &= y_1 - 3y_2 + u(t-1)e^t \\
y_1(0) &= 3 \\
y_2(0) &= 0
\end{aligned}
$$

First we do the Laplace transform on the two equations:

$$
\begin{aligned}
sY_1 - 3 &= 2Y_1 - 4Y_2 + \frac{e^{-s+1}}{s-1} \\
sY_2 - 0 &= Y_1 - 3Y_2 + \frac{e^{-s+1}}{s-1}
\end{aligned}
$$

This must be solved algebraically for $Y_1(s)$ and $Y_2(s)$

$$
\begin{aligned}
(s-2)Y_1 &= 3 - 4Y_2 + \frac{e^{-s+1}}{s-1} \\
(s+3)Y_2 &= Y_1 + \frac{e^{-s+1}}{s-1}
\end{aligned}
$$

We insert $Y_1 = \frac{3}{s-2} - \frac{4Y_2}{s-2} + \frac{e^{-s+1}}{(s-2)(s-1)}$ into the exspression for Y_2:

$$
\begin{aligned}
(s+3)Y_2 &= \frac{3}{s-2} - \frac{4Y_2}{s-2} + \frac{e^{-s+1}}{(s-2)(s-1)} + \frac{e^{-s+1}}{s-1} \\
(s-2)(s+3)Y_2 &= 3 - 4Y_2 + \frac{e^{-s+1}}{s-1} + (s-2)\frac{e^{-s+1}}{s-1} \\
(s^2+s-2)Y_2 &= 3 + (s-1)\frac{e^{-s+1}}{s-1} \\
(s+2)(s-1)Y_2 &= 3 + e^{-s+1} \\
Y_2 &= \frac{3}{(s+2)(s-1)} + e\frac{e^{-s}}{(s+2)(s-1)}
\end{aligned}
$$

Partial fractions:

$$\frac{1}{(s+2)(s-1)} = \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right)$$

Then:

$$\begin{aligned}
y_2 &= \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+2} + \frac{ee^{-s}}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right)\right\} \\
y_2 &= e^t - e^{-2t} + \frac{e}{3}u(t-1)\left(e^{t-1} - e^{-2t+2}\right) \\
y_2 &= e^t - e^{-2t} + \frac{1}{3}u(t-1)\left(e^{t} - e^{3-2t}\right)
\end{aligned}$$

And for y_1:

$$Y_1 = \frac{3}{s-2} - \frac{12}{(s^2-4)(s-1)} - e\frac{4e^{-s}}{(s^2-4)(s-1)} + e\frac{e^{-s}}{(s-2)(s-1)}$$

Partial fractions:

$$\frac{12}{(s^2-4)(s-1)} = \frac{4s+4}{s^2-4} - \frac{4}{s-1}$$

$$\frac{1}{(s-2)(s-1)} = \frac{1}{s-2} - \frac{1}{s-1}$$

$$\begin{aligned}
y_1 &= 3e^{2t} - 4\cosh 2t - 2\sinh 2t + 4e^t - e\mathcal{L}^{-1}\left\{e^{-s}\left(\frac{4}{(s^2-4)(s-1)} - \frac{1}{(s-2)(s-1)}\right)\right\} \\
y_1 &= -e^{-2t} + 4e^t - e\mathcal{L}^{-1}\left\{e^{-s}\left(\frac{4s+4}{3(s^2-4)} - \frac{4}{3(s-1)} - \frac{1}{s-2} + \frac{1}{s-1}\right)\right\} \\
y_1 &= -e^{-2t} + 4e^t - e\mathcal{L}^{-1}\left\{e^{-s}\left(\frac{4s+4}{3(s^2-2^2)} - \frac{1}{3(s-1)} - \frac{1}{s-2}\right)\right\} \\
y_1 &= -e^{-2t} + 4e^t - e\frac{1}{3}u(t-1)\left(4\cosh(2t-2) + 2\sinh(2t-2) - e^t - 3e^{2t}\right) \\
y_1 &= -e^{-2t} + 4e^t - e\frac{1}{3}u(t-1)\left(2e^{2t-2} + 2e^{-2t+2} + e^{2t-2} - e^{-2t+2} - e^{t-1} - 3e^{2t-2}\right) \\
y_1 &= -e^{-2t} + 4e^t - \frac{1}{3}u(t-1)\left(2e^{2t-1} + 2e^{-2t+3} + e^{2t-1} - e^{-2t+3} - e^{t} - 3e^{2t-1}\right) \\
y_1 &= -e^{-2t} + 4e^t + \frac{1}{3}u(t-1)\left(-e^{3-2t} + e^t\right)
\end{aligned}$$

10.2.6 We are to find the Fourier series of $f(x) = x$, for $(0 < x < 2\pi)$. If we want, we may move the function to be centered around zero: $f(x) = x$ for $(0 < x < \pi)$ and $f(x) = x + 2\pi$ for $(-\pi < x < 0)$. We then find the Fourier coefficients:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \pi$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{0}(x+2\pi)\cos nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\mathrm{d}x + 2\int_{-\pi}^{0}\cos nx\mathrm{d}x \\
&= \frac{1}{\pi}\left(\frac{1}{n^2}\cos nx + \frac{x}{n}\sin nx\right)_{-\pi}^{\pi} + \frac{2}{n}\left(\sin nx\right)_{-\pi}^{0} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0}(x+2\pi)\sin nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\mathrm{d}x + 2\int_{-\pi}^{0}\sin nx\mathrm{d}x \\
&= \frac{-2}{n} + \frac{2(-1)^n}{n}
\end{aligned}$$

Then the Fourier series is:

$$\pi + \sum_{n=1}^{\infty}\left(\left(\frac{-2}{n} + \frac{2(-1)^n}{n}\right)\sin nx\right)$$

10.2.13 We are to find the Fourier series of $f(x) = 1$ for $(-\pi/2 < x < \pi/2)$ and $f(x) = -1$ for $(\pi/2 < x < 3\pi/2)$.

a_0 equals the mean over one periode, and no integrals are necessary:

$$a_0 = 0$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos nx\mathrm{d}x - \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}\cos nx\mathrm{d}x \\
a_n &= \frac{2}{\pi n}\left(\sin\frac{\pi n}{2}\right) - \frac{1}{\pi n}\sin\frac{3n\pi}{2} + \frac{1}{\pi n}\sin\frac{n\pi}{2} \\
a_n &= \frac{4}{\pi n}(-1)^{\frac{n-1}{2}} \quad n = 1, 3, 5\ldots
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin nx\mathrm{d}x - \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}\sin nx\mathrm{d}x \\
b_n &= \frac{1}{\pi n}\cos\frac{3n\pi}{2} - \frac{1}{\pi n}\cos\frac{n\pi}{2} \\
b_n &= 0
\end{aligned}$$

Then the Fourier series is:

$$\sum_{n=1,3,5}^{\infty}\left(\frac{4}{\pi n}(-1)^{\frac{n-1}{2}}\cos nx\right)$$

10.2.15 We are to find the Fourier series of $f(x) = x$ for $(-\pi/2 < x < \pi/2)$ and $f(x) = 0$ for $(\pi/2 < x < 3\pi/2)$.

a_0 equals the mean over one periode, and again no integrals are necessary:

$$a_0 = 0$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos nx\mathrm{d}x \\
a_n &= \frac{1}{\pi}\left(\frac{1}{n^2}\cos nx + \frac{x}{n}\sin nx\right)_{-\pi/2}^{\pi/2} \\
a_n &= \frac{1}{\pi}\left(\frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\cos\frac{-n\pi}{2} + \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\sin\frac{-n\pi}{2}\right) \\
a_n &= 0
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\mathrm{d}x \\
b_n &= \frac{1}{\pi}\left(\frac{1}{n^2}\sin nx - \frac{x}{n}\cos nx\right)_{-\pi/2}^{\pi/2} \\
b_n &= \frac{1}{\pi}\left(\frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{-n\pi}{2} - \frac{\pi}{2n}\cos\frac{n\pi}{2} - \frac{\pi}{2n}\cos\frac{-n\pi}{2}\right) \\
b_n &= \frac{1}{\pi}\left(\frac{2}{n^2}\sin\frac{n\pi}{2} - \frac{\pi}{n}\cos\frac{n\pi}{2}\right) \\
b_n &= \frac{2}{\pi n^2}(-1)^{\frac{n-1}{2}} \quad n = 1,3,5\ldots \\
b_n &= \frac{\pi}{n}(-1)^{n+1} \quad n = 2,4,6\ldots
\end{aligned}$$

Then the Fourier series is:

$$\frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x + \ldots$$

10.3.9 We are to find the Fourier series to the periodic function $f(x)$ given by

$$f(x) = 0 \text{ for } -1 < x < 0,\ f(x) = x \text{ for } 0 < x < 1,\ \text{periode} p = 2L = 2$$

With partial integration:

$$
\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{1} f(x)\mathrm{d}x = \frac{1}{4} \\
a_n &= \int_{-1}^{1} f(x)\cos n\pi x\mathrm{d}x = \int_{0}^{1} x\cos n\pi x\mathrm{d}x = \left|\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right|_0^1 = \frac{\cos n\pi - 1}{n^2\pi^2} \\
b_n &= \int_{-1}^{1} f(x)\sin n\pi x\mathrm{d}x = \int_{0}^{1} x\sin n\pi x\mathrm{d}x = \left|-\frac{x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_0^1 = \frac{-\cos n\pi}{n\pi} \\
&= \frac{-(-1)^n}{n\pi} = \frac{(-1)^{m+1}}{n\pi}
\end{aligned}
$$

We see that $a_n = 0$ for $n = 2, 4, \ldots$, and $a_n = -2/n^2\pi^2$ for $n = 1, 3, \ldots$.
Then the fourier series is:

$$
\frac{1}{4} - \frac{2}{\pi^2}\left(\cos\pi x + \frac{1}{9}\cos 3\pi x + \cdots\right) + \frac{1}{\pi}\left(\sin\pi x - \frac{1}{2}\sin 2\pi x + - \cdots\right)
$$