Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 5

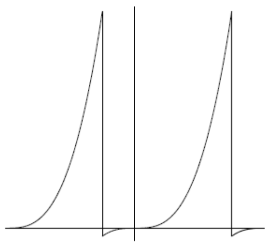
$\boxed{10.4.2}$ Check if $f(-x) = f(x)$ (even) or $f(-x) = -f(x)$ (odd)

Even functions: $|x|$, e^{x^2}, $\sin^2 x$, $x \sin x$ and $e^{-|x|}$
Odd functions: $x \cos x$

$\boxed{10.4.9}$ We are to determine if $f(x)$ given by

$$f(x) = x^3 \quad \text{for} \quad -\pi/2 < x < 3\pi/2, \quad f(x + 2\pi) = f(x)$$

are even (symmetric about the y-axis) or odd (symmetric about origo), or none. When we draw the function on $-2\pi < x < 2\pi$, we see that it is neither.



$\boxed{10.4.15}$ We are to find the Fourier series of the 2π-periodic function $f(x) = x^2/2$, $-\pi < x < \pi$. Since $f(x)$ is even, this is the Fourier cosine series:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^\pi \frac{x^2}{2}\mathrm{d}x = \frac{\pi^2}{6} \\ a_n &= \frac{2}{\pi}\int_0^\pi \frac{x^2}{2}\cos nx \mathrm{d}x \end{aligned}$$

Use then Rottmann page 144 formula 124:

$$a_n = \frac{1}{\pi}\left[\frac{2}{n^2}x\cos nx - \frac{2 - n^2x^2}{n^3}\sin nx\right]_0^\pi = \frac{2\cos n\pi}{n^2} = 2\frac{(-1)^n}{n^2}, \quad n \geq 1$$

Then the Fourier cosine series are:

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$\boxed{10.4.18}$ We are to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

From excercis 15:

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

Since $f(\pi) = \pi^2/2$ we have:

$$f(\pi) = \frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

and then is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2/2 - \pi^2/6}{2} = \frac{\pi^2}{6}$$

$\boxed{10.4.23}$ We are to find the cosine and the sine series of the function $f(x) = \pi - x$, $0 < x < \pi$.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\mathrm{d}x = \frac{1}{\pi} \cdot \frac{-1}{2} \left|(\pi - x)^2\right|_0^{\pi} = \frac{\pi}{2}$$

and by partial integration:

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\mathrm{d}x = \frac{2}{\pi}\left[\left|(\pi - x)\frac{\sin nx}{n}\right|_0^{\pi} + \int_0^{\pi} \frac{\sin nx}{n}\mathrm{d}x\right] = \frac{2(1 - \cos nx)}{n^2\pi}$$

When n is even $\cos n\pi = 1$ and $a_n = 0$. When n is odd $\cos n\pi = -1$ and $a_n = 4/(n^2\pi)$. Then we have the series:

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}cos3x + \frac{1}{5^2}cos5x + \cdots\right) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}$$

For the sine series, we get:

$$b_n = \frac{2}{\pi}\int_0^{\infty} (\pi - x)\sin nx\mathrm{d}x = \frac{2}{\pi}\left[-\left|(\pi - x)\frac{\cos nx}{n}\right|_0^{\pi} - \underbrace{\int_0^{\pi} \frac{\cos nx}{n}\mathrm{d}x}_{0}\right] = \frac{2}{\pi}$$

$\boxed{10.5.5}$ Look at lfov5_2.*pdf*...

$\boxed{10.5.7}$ Look at lfov5_2.*pdf*...

$\boxed{10.6.4}$ We are to find a general solution of the differential equation $\ddot{y} + \omega^2 \dot{y} = r(t)$ with

$$r(t) = \cos \alpha t + cos \beta t \quad \omega^2 \neq \alpha^2, \beta^2$$

We know that this solution has the form $y = y_h + y_p$, where y_h is a general solution of $\ddot{y} + \omega^2 \dot{y} = 0$ and y_p is a particular solution of $\ddot{y} + \omega^2 \dot{y} = r(t)$.
The caracteristic equation for $\ddot{y} + \omega^2 \dot{y} = 0$ is $\lambda^2 + \omega^2 = 0$ with roots $\lambda = \pm i\omega$. Then we know that:

$$y_h = C_1 \cos \omega t + C_2 \sin \omega t$$

A particular solution of $\ddot{y} + \omega^2 \dot{y} = r(t)$ has the form (Method of undetermined coefficients, Kreyszig p. 105):

$$y_p = A_\alpha \cos \alpha t + B_\beta \cos \beta t + C_\alpha \sin \alpha t + D_\beta \sin \beta t$$

Then we insert $y = y_p$ into $\ddot{y} + \omega^2 \dot{y} = r(t)$:

$$\begin{aligned} -\alpha^2 A_\alpha \cos \alpha t - \beta^2 B_\beta \cos \beta t - \\ \alpha^2 C_\alpha \sin \alpha t - \beta^2 D_\beta \sin \beta t + \\ \omega^2 A_\alpha \cos \alpha t + \omega^2 B_\beta \cos \beta t + \\ \omega^2 C_\alpha \sin \alpha t + \omega^2 D_\beta \sin \beta t - \\ = \cos \alpha + \cos \beta \end{aligned}$$

We look at the coefficients before $\cos \alpha t$, $\cos \beta t$, $\sin \alpha t$ and $\sin \beta t$ and find that:

$$\begin{aligned} (-\alpha^2 + \omega^2) A_\alpha = 1 \Rightarrow A_\alpha &= \frac{1}{\omega^2 - \alpha^2} \\ (-\beta^2 + \omega^2) B_\beta = 1 \Rightarrow B_\beta &= \frac{1}{\omega^2 - \beta^2} \\ (-\alpha^2 + \omega^2) C_\alpha = 0 \Rightarrow C_\alpha &= 0 \\ (-\beta^2 + \omega^2) D_\beta = 0 \Rightarrow D_\beta &= 0 \end{aligned}$$

Then the general solution is

$$y = y_h + y_p = C_1 \cos \omega t + C_2 \sin \omega t + \frac{1}{\omega^2 - \alpha^2} \cos \alpha t + \frac{1}{\omega^2 - \beta^2} \cos \beta t$$

$\boxed{10.6.5}$ Look at lfov5$_2$.*pdf*...

$\boxed{10.7.7}$ Look at lfov5$_2$.*pdf*...

$\boxed{10.7.12}$ Look at lfov5$_3$.*pdf*...