TMA4125
Matematikk 4N
våren 2004

Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag

Løsningsforslag - Øving 7

Fra Kreyszig, avsnitt 10.7

7 Funksjonen $f(x)$ er odde så Fourierrekka blir ei sinusrekke.

Ved å bruke formel 121 i Rottmann s. 144 (eller delvis integrasjon) finn vi

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx \\ &= \frac{2}{\pi}\left[\frac{1}{n^2}\sin nx - \frac{x}{n}\cos nx\right]_0^{\pi/2} = \frac{2}{\pi}\left[\frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{\pi}{2n}\cos\frac{n\pi}{2}\right]. \end{aligned}$$

Dette gir

$$\begin{aligned} b_{2m-1} &= \frac{2}{\pi}\frac{\sin(m-\frac{1}{2})\pi}{(2m-1)^2} = \frac{2(-1)^{m-1}}{\pi(2m-1)^2} \qquad (m=1,2,3,\dots) \\ b_{2m} &= \frac{2}{\pi}\left[-\frac{\pi\cos m\pi}{4m}\right] = \frac{(-1)^{m+1}}{2m} \qquad (m=1,2,3,\dots). \end{aligned}$$

Rekkeutviklinga blir derfor (merk at $(-1)^{m-1} = (-1)^{m+1}$):

$$\begin{aligned} f(x) &\sim \sum_{m=1}^{\infty}(-1)^{m+1}\left[\frac{2}{\pi(2m-1)^2}\sin(2m-1)x + \frac{1}{2m}\sin 2mx\right] \\ &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x + \cdots. \end{aligned}$$

Vi har altså

$$b_1 = \frac{2}{\pi}, \quad b_2 = \frac{1}{2}, \quad b_3 = -\frac{2}{9\pi}, \quad b_4 = -\frac{1}{4}, \quad b_5 = \frac{2}{25\pi}.$$

Kvadratfeilen får sitt minimum E_N^* for Fourierpolynomet $F_N(x) = \sum_{n=1}^{N} b_n \sin nx$. Altså

$$\begin{aligned} F_1(x) &= \frac{2}{\pi}\sin x \\ F_2(x) &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x \\ F_3(x) &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x \\ F_4(x) &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x \\ F_5(x) &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x. \end{aligned}$$

Etter formel (6) i Kreyszig 10.7 har vi

$$E_N^* = \int_{-\pi}^{\pi} f(x)^2\,dx - \pi \sum_{n=1}^{N} b_n^2.$$

Her er $\int_{-\pi}^{\pi} f(x)^2\,dx = \int_{-\pi/2}^{\pi/2} x^2\,dx = \pi^3/12$, og vi legg merke til at $E_{N+1}^* = E_N^* - \pi b_{N+1}^2$. Vi får

$$E_1^* = \pi^3/12 - \pi b_1^2 \approx 1.311, \qquad E_2^* = E_1^* - \pi b_2^2 \approx 0.525, \qquad E_3^* = E_2^* - \pi b_3^2 \approx 0.509$$
$$E_4^* = E_3^* - \pi b_4^2 \approx 0.313, \qquad E_5^* = E_4^* - \pi b_5^2 \approx 0.311.$$

$\boxed{11}$ Fra Kreyszig 10.2 oppgave 7 med fasit har vi

$$\begin{aligned} x^2 &= \frac{\pi^2}{3} - 4\Big(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots\Big) \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx, \qquad -\pi \le x \le \pi. \end{aligned}$$

Parsevals identitet, Kreyszig 10.7 formel (8), gir da

$$2\left(\frac{\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty}\left(4\frac{(-1)^n}{n^2}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(x^2\right)^2\,dx,$$
$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{16}\left(2\cdot\frac{\pi^4}{5} - 2\cdot\frac{\pi^4}{9}\right) = \frac{\pi^4}{90} \qquad (\approx 1.082).$$

De fire første partialsummene er

$$S_1 = 1, \quad S_2 = S_1 + 1/2^4 \approx 1.063, \quad S_3 = S_2 + 1/3^4 \approx 1.075, \quad S_4 = S_3 + 1/4^4 \approx 1.079.$$

Fra Kreyszig, avsnitt 10.8

$\boxed{2}$ Vi skal bruke Fourierintegral til vise at

$$\int_0^{\infty} \frac{\sin w \cos wx}{w}\,dw = \begin{cases} \pi/2 & \text{hvis } 0 \le x < 1 \\ \pi/4 & \text{hvis } x = 1 \\ 0 & \text{hvis } x > 1. \end{cases} \tag{$*$}$$

Av svaret ser vi at vi skal betrakte en jevn funksjon $f(x)$ som oppfyller $f(x) = \pi/2$ for $0 < x < 1$ og $f(x) = 0$ for $x > 1$. Fourierintegralet til $f(x)$ blir et cosinusintegral ((11) Kreyszig s. 562) der

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos wx\,dx = \int_0^1 \cos wx\,dx = \frac{\sin wx}{w}\bigg|_0^1 = \frac{\sin w}{w}.$$

Flgelig kan $f(x)$ representeres ved Forierintegralet

$$f(x) = \int_0^{\infty} \frac{\sin w}{w}\cos wx\,dw. \tag{1}$$

For $x = 1$ er $f(x)$ diskontinuerlig, og middelverdien av grenseverdien for $f(x)$ fra hyre (0) og venstre ($\pi/2$) er $\pi/4$. Dermed flger $(*)$ av (1) og teoremet om Fourierintegral (Kreyszig s. 559).

$\boxed{15}$ Gitt $f(x) = \sin x$ for $0 < x < \pi$, 0 for $x > \pi$. Vi skal finne Fouriersinusintegralet til $f(x)$. Fra (12) Kreyszig s. 562 får vi, når vi bruker den trigonometriske formelen $2\sin A \sin B = \cos(A - B) - \cos(A + B)$ (Rottmann s. 88),

$$\begin{aligned}
B(w) &= \frac{2}{\pi}\int_0^\infty f(x)\sin wx\,dx = \frac{2}{\pi}\int_0^\pi \sin x \sin wx\,dx \\
&= \frac{1}{\pi}\int_0^\pi [\cos(1-w)x - \cos(1+w)x]\,dx = \frac{1}{\pi}\Big[\frac{\sin(1-w)x}{1-w} - \frac{\sin(1+w)x}{1+w}\Big]_0^\pi \\
&= \frac{1}{\pi}\Big[\frac{\sin(1-w)\pi}{1-w} - \frac{\sin(1+w)\pi}{1+w}\Big] = \frac{1}{\pi}\Big[\frac{\sin \pi w}{1-w} + \frac{\sin \pi w}{1+w}\Big] = \frac{2\sin \pi w}{\pi(1-w^2)}
\end{aligned}$$

idet $\sin(1-w)\pi = \sin(\pi - \pi w) = \sin \pi w$ og $\sin(1+w)\pi = \sin(\pi w + \pi) = -\sin \pi w$.

Dermed blir Fouriersinusintegralet ((13) Kreyszig s. 562)

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin \pi w}{1-w^2}\sin wx\,dw$$

dvs.

$$\int_0^\infty \frac{\sin \pi w}{1-w^2}\sin wx\,dw = \begin{cases} \pi(\sin x)/2 & \text{for } 0 \le x \le \pi \\ 0 & \text{for } x > \pi. \end{cases}$$

Fra Kreyszig, avsnitt 10.10

$\boxed{7}$ Vi skal finne den Fouriertransformerte av

$$f(x) = \begin{cases} xe^{-x} & \text{for } x > 0, \\ 0 & \text{for } x < 0 \end{cases}$$

Ved bruk av definisjonen av Fouriertransformasjonen (ligning (6), side 570 i Kreyszig(8. utg.)) og delvis integrasjon finner vi

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^\infty xe^{-x}e^{-iwx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{-1-iw}xe^{-(1+iw)x} - \frac{1}{(-1-iw)^2}e^{-(1+iw)x}\right]_0^\infty = \frac{1}{\sqrt{2\pi}(1+iw)^2}.
\end{aligned}$$

Kommentar til innsetting av grensene i utregningen over:

$$xe^{-x}e^{-iwx} = xe^{-x}(\cos\omega x - i\sin\omega x) \underset{x\to\infty}{\longrightarrow} 0,$$

idet $xe^{-x} = x/e^x \to 0$ og $\sin\omega x$ og $\cos\omega x$ er begrenset.

$\boxed{10}$ Gitt $f(x) = |x|$ for $|x| < 1$, 0 for $|x| > 1$. Vi søker den Fouriertransformerte til $f(x)$.

$$\begin{aligned}
\mathcal{F}\{f(x)\} = \widehat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} |x| e^{-iwx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} |x|(\cos wx - i \sin wx)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\underbrace{|x| \cos wx}_{\text{jevn funksjon}} - i \underbrace{|x| \sin wx}_{\text{odde funksjon}})\, dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{1} x \cos wx\, dx \\
&= \sqrt{\frac{2}{\pi}} \left[x \frac{\sin wx}{w} + \frac{\cos wx}{w^2} \right]_0^1 = \sqrt{\frac{2}{\pi}} \left[\frac{\sin w}{w} + \frac{\cos w - 1}{w^2} \right]
\end{aligned}$$

Alternativt kan vi innføre $|x| = x$ for $x > 0$, $|x| = -x$ for $x < 0$:

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} |x| e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} (-x) e^{-iwx}\, dx + \int_{0}^{1} x e^{-iwx}\, dx \right] = \dots$$

Et alternativt svar (som fes ved delvis integrasjon av eksponentialfunksjon) er:

$$\widehat{f}(w) = \frac{-1}{w^2} + \left(\frac{1}{w^2} - \frac{1}{iw} \right) \cosh(iw)$$

Fra SIF5013 mai 2000

$\boxed{3}$ **a)**

$$x = 0:$$

$$\frac{\sinh \pi}{\pi} [\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}] = f(0) = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{2 \sinh \pi} - \frac{1}{2}$$

$$x = \pi:$$

$$\frac{2 \sinh \pi}{\pi} [\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} (-1)^n] = \frac{f(\pi+0) + f(\pi-0)}{2} = \frac{e^{\pi} + e^{-\pi}}{2} = \cosh \pi$$

Følgelig: $\displaystyle \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^2 n}{1+n^2} = \frac{\pi}{2 \sinh \pi} \cdot \cosh \pi$ dvs. $\displaystyle \sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi \cosh \pi}{2 \sinh \pi} - \frac{1}{2}$

b)

$$\text{Invers Fouriertransformert:} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\cos(\pi w/2)}{1-w^2}e^{iwx}dw = f(x) \text{ for alle } x$$

$$x = 0: \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}dw = f(0) = 1 \quad \text{dvs.} \quad \int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}dw = \pi$$

$$x = \pi/2: \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}e^{i\pi w/2}dw = f(\pi/2) = 0$$

$$\text{Eulers formel gir} \int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}\cos\frac{\pi w}{2} + i\int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}\sin\frac{(\pi w/2)}{2}dw = 0$$

$$\text{og følgelig:} \int_{-\infty}^{\infty}\frac{\cos^2(\pi w/2)}{1-w^2}dw = 0$$

Fra SIF5016 desember 2000

$\boxed{1}$ Fra den komplekse Fourierrekka kan vi finne den vanlige Fourierrekka,

$$\begin{aligned}
f(x) &\sim \frac{\sinh\pi}{\pi}\Big[\sum_{n=-1}^{-\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx} + 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx}\Big] \\
&= \frac{\sinh\pi}{\pi}\Big[\sum_{n=1}^{\infty}(-1)^n\frac{1-in}{1+(-n)^2}e^{-inx} + 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx}\Big] \\
&= \frac{\sinh\pi}{\pi}\Big[1 + \sum_{n=1}^{\infty}(-1)^n\frac{(1-in)e^{-inx}+(1+in)e^{inx}}{1+n^2}\Big] \\
&= \frac{2\sinh\pi}{\pi}\Big[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx - n\sin nx)\Big]
\end{aligned}$$

Setter inn $x = 0$:

$$1 = \frac{2\sinh\pi}{\pi}\Big[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(1-0)\Big] \quad \text{som gir} \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{\pi}{2\sinh\pi} - \frac{1}{2}$$

Setter inn $x = \pi$ og bruker at $\lim_{x\to\pi-} f(x) = e^{\pi}, \lim_{x\to\pi+} f(x) = e^{-\pi}$:

$$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{2\sinh\pi}{\pi}\Big[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cdot(-1)^n\Big] \quad \text{som gir} \quad \sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{\pi\cosh\pi}{2\sinh\pi} - \frac{1}{2}$$