Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4125 Matematikk 4N
Spring 2006

Solutions to exercise set 8

$\boxed{11.5.15}$ We have the heat equation ($c = 1$):

$$u_t = u_{xx}$$

with the initial conditions:

$$\begin{aligned} -u_x(\pi, t) &= u(\pi, t) \\ u(0, t) &= 0 \end{aligned}$$

When we set $u(x, t) = F(x)G(t)$ this becomes:

$$\begin{aligned} F(X)\dot{G}(T) &= F''(X)G(T) \\ \frac{F''}{F} &= \frac{\dot{G}}{G} = k \end{aligned}$$

with the initial conditions:

$$\begin{aligned} -F'(\pi)G(t) &= F(\pi)G(t) \to -F'(\pi) = F(\pi) \\ F(0)G(t) &= 0 \to F(0) = 0 \end{aligned}$$

We then solves $F'' - kF = 0$:

Case 1: $k = 0$

$F'' = 0$ has the general solution:

$$F = ax + b$$

Using the initial conditions, we find that $a = b = 0$, and this case is therefore of no interest to us.

Case 2: $k = p^2$

$F'' - p^2 F = 0$ has the general solution:

$$F = Ae^{px} + Be^{-px}$$

Using $F(0) = 0$, we find

$$\begin{aligned} F(0) &= A + B = 0 \\ A &= -B \end{aligned}$$

Then we use $-F'(\pi) = F(\pi)$:

$$\begin{aligned} -F'(\pi) = -Ape^{p\pi} - Ape^{-p\pi} &= Ae^{p\pi} - Ae^{-p\pi} \\ A(1-p)e^{-p\pi} &= A(1+p)e^{p\pi} \\ A &= \left(\frac{1+p}{1-p} e^{2p\pi} \right) A \end{aligned}$$

Since the expression in parenthesis is negative for large p, $A = 0$, and this case is therefore of no interest to us.

Case 3: $k = p^2$

$F'' + p^2 F = 0$ has the general solution:

$$F = A \cos px + B \sin px$$

Using $F(0) = 0$, we find

$$F(0) = A = 0$$

Then we use $-F'(\pi) = F(\pi)$ we find:

$$\begin{aligned} -Br \cos px &= B \sin px \\ -p &= \tan px \end{aligned}$$

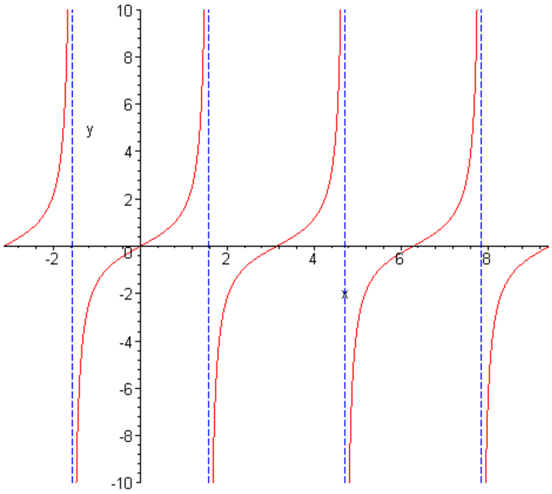
Then we have that $F(x) = B \sin px$, where p is a solution of $-p = \tan p$.

We then solve $\dot{G} + p^2 G = 0$, that has the general solution:

$$G(t) = Ce^{-p^2 t}$$

We then set $BC = 1$, and have that $\sin pxe^{-p^2 t}$, where $-p = \tan \pi p$, is a solution of our heat equation.

To show that $-p = \tan \pi p$ has infinitely many positive solutions, we look at the graph of $\tan x$:

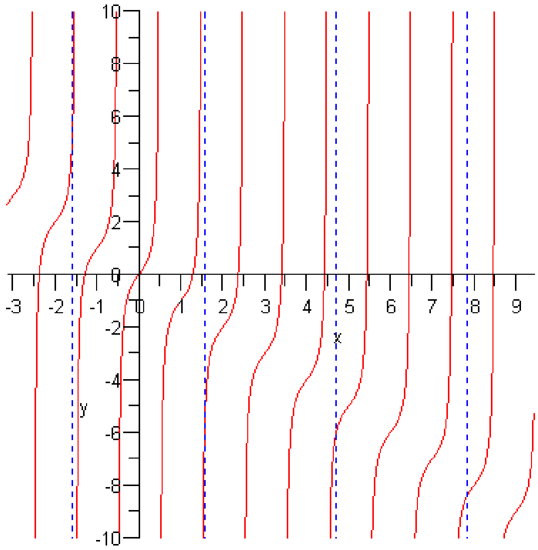


The function

$$\tan \pi p - p$$

is just a skewed version of this graph, the basic is the same: *In every period, the graph takes all the values from* $-\infty$ *to* ∞. Since the function is continous in every period, it has to cross zero in every period. These crossings are $p_1, p_2, \ldots$.

The exact graph is:



$\tan x$ is periodic with period π, and $\tan \pi p$ has period 1, on the interval $[n - \frac{1}{2}, n + \frac{1}{2}]$ for $n \in \mathcal{N}$.

Since the crossing is within the period, it follows that $\tan \pi p_n - p_n > n - \frac{1}{2}$.

We can see from the graph that, because we are subtracting p from $\tan \pi p$, the crossings moves to the right in each period. This is a graphical way to say that $\lim_{n\to\infty}(p_n - n + \frac{1}{2}) = 0$.