Crank - Nikolson metode

$$u_t = u_{xx}$$

Vi tar utgangspunktet i punktet $P_{i,j+\frac{1}{2}}$

$$P_{i,j+\frac{1}{2}} = \left(x_i, \; _j + \frac{k}{2}\right)$$

der $x_i = ih$, $t_j = jk$.

Taylors utvikling i $(x, t+\frac{k}{2})$ gir

$$① \quad u(x, t+k) = u(x, t+\tfrac{k}{2}) + \tfrac{k}{2} u_t(x, t+\tfrac{k}{2}) + \left(\tfrac{k}{2}\right)^2 \frac{u_{tt}(x, t+\frac{k}{2})}{2} + O(k^3)$$

og

$$② \quad u(x, t) = u(x, t+\tfrac{k}{2}) - \tfrac{k}{2} u_t(x, t+\tfrac{k}{2}) + \left(\tfrac{k}{2}\right)^2 u_{tt}(x, t+\tfrac{k}{2}) + O(k^3)$$

Ta $x = x_i$ og $t = t_j$, vi skriver differensen ① - ② og får

$$u(x_i, t_j+k) - u(x_i, t_j) = k\, u_t(x_i, t_j+\tfrac{k}{2}) + O(k^3)$$

Dermed

$$u_t(P_{i,j+\frac{1}{2}}) = u_t(x_i, t_j+\tfrac{k}{2}) = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k^2) \quad ③$$

Vi bruker den følgende approksimasjonen:

$$u_t(P_{i,j+\frac{1}{2}}) \approx \frac{u_{i,j+1} - u_{i,j}}{k}$$

Dette er en approksimasjon av 2. orden. (siden vi har $O(k^2)$ i ③) som er bedre enn for den forrigje metoden som hadde orden 1.

For $u_{xx}(P_{i,j+\frac{1}{2}})$ bruker vi

$$u_{xx}(P_{i,j+\frac{1}{2}}) \approx \frac{1}{2}\left[\underbrace{\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2}}_{\approx\, u_{xx}(P_{i,j})} + \underbrace{\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}}_{\approx\, u_{xx}(P_{i,j+1})}\right]$$

Dette er også en 2. orden approksimasjon

Ligningen

$$u_t = u_{xx}$$

medfører

$$u_t(P_{i,j+\frac{1}{2}}) = u_{xx}(P_{i,j+\frac{1}{2}}) \, .$$

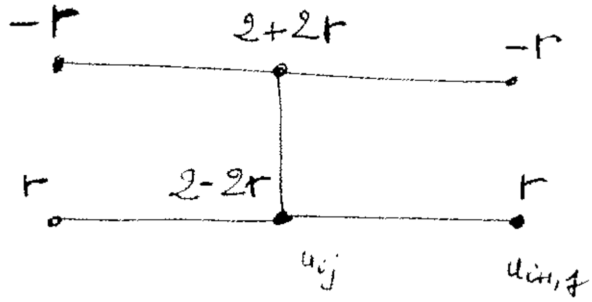
Derfor får vi den følgende diskretiseringen for ligningen

$$\frac{u_{i,j+1}-u_{ij}}{k} = \frac{1}{2h^2}\left[u_{i+1,j}-2u_{ij}+u_{i-1,j}+u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}\right]$$

$$\Longleftrightarrow \quad u_{i,j+1}-u_{ij} = \frac{r}{2}\left[u_{i+1,j}-2u_{ij}+u_{i-1,j}+u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}\right]$$

$$\Longleftrightarrow \quad \boxed{-r\,u_{i-1,j+1}+(2+2r)\,u_{i,j+1}-r\,u_{i+1,j+1} = r\,u_{i+1,j}+(2-2r)\,u_{i,j}+r\,u_{i-1,j}} \qquad (4)$$

"Stencil"



Vektor notasjon:

La $\mathbf{u}_j = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N-1,j} \end{pmatrix}$

Ligning ④ for $i = 1, \dots, N-1$ er ekvivalente til

$$A\,\mathbf{u}_{j+1} = B\,\mathbf{u}_j + \mathbf{b}_j \qquad (4')$$

der

$$A = \begin{pmatrix} 2+2r & -r & & 0 \\ -r & \ddots & \ddots & \\ & \ddots & \ddots & -r \\ 0 & & -r & 2+2r \end{pmatrix}$$

$$B = \begin{pmatrix} 2-2r & r & & 0 \\ r & \ddots & \ddots & \\ & \ddots & \ddots & r \\ 0 & & r & 2-2r \end{pmatrix}$$

$$\mathbf{b}_j = \begin{pmatrix} r u_{0,j} + r u_{0,j+1} \\ 0 \\ \vdots \\ 0 \\ r u_{N,j} + r u_{N,j+1} \end{pmatrix} = \begin{pmatrix} r g_1(jk) + r g_1((j+1)k) \\ 0 \\ \vdots \\ 0 \\ r g_2(jk) + r g_2((j+1)k) \end{pmatrix}$$

Eksempel

$$u_t = u_{xx} \quad \text{for } x \in [0,1]$$

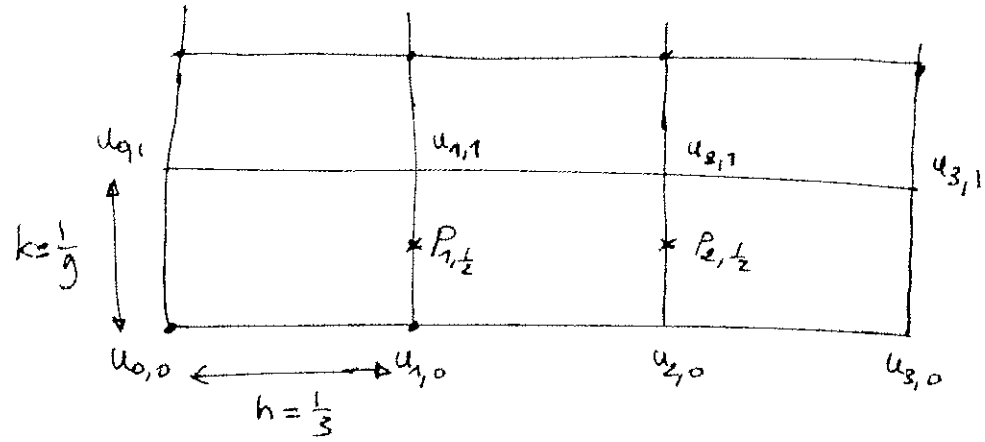
$$u(x,0) = \sin(\pi x) \qquad \text{startsbetingelse}$$

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad \text{randbetingelse}$$

Diskretisering med $N = 3$ (det vil si $h = \frac{1}{3}$ siden $L = 1$)

~~og~~ $k = \frac{1}{9}$.

Da har man $r = \frac{k}{h^2} = 1$.



For får man fra ligning ④ at

$$-r \underbrace{u_{0,1}}_{\substack{= \\ 0}} + \underbrace{(2+2r)}_{=4} u_{1,1} - r u_{2,1} = r \underbrace{u_{0,0}}_{\substack{= \\ 0}} + \underbrace{(2-2r)}_{=0} u_{1,0} + r \underbrace{u_{2,0}}_{\substack{= \\ f(2h) \\ = \\ \sin\left(\frac{2\pi}{3}\right)}}$$

(fra randbetingelse) (fra randbet.) (startsbeting.)

dermed

$$4u_{1,1} - u_{2,1} = \sin\left(\frac{2\pi}{3}\right)$$

For får man fra ligning (4) at

$$-r\,\underbrace{u_{1,1}}\; + (2+2r)\,u_{2,1} - r\,\underbrace{u_{3,1}}_{\substack{= \\ 0 \\ \text{(fra rand-} \\ \text{betingelse)}}} = r\,\underbrace{u_{1,0}}_{\substack{= \\ f(h) \\ = \\ \sin\left(\frac{\pi}{3}\right) \\ \text{(fra starts} \\ \text{betingelse}}} + \underbrace{(2-2r)}_{=0}\,u_{2,0} + r\,\underbrace{u_{3,0}}_{\substack{=0 \\ \text{(fra rand-} \\ \text{betingelse)}}}$$

dermed,

$$-u_{1,1} + 4u_{2,1} = \sin\left(\frac{\pi}{3}\right)$$

Vi må løse

$$\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} u_{1,1} \\ u_{2,1} \end{pmatrix} = \begin{pmatrix} \sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) \end{pmatrix} \qquad (5)$$

Vi får

$$\begin{pmatrix} u_{1,1} \\ u_{2,1} \end{pmatrix} \approx \begin{pmatrix} 0{,}2887 \\ 0{,}2887 \end{pmatrix}$$

Vi er ferdig med det 1. tidsskrittet;

$$u\left(\tfrac{1}{3}, \tfrac{1}{9}\right) \approx u_{1,1} = 0{,}2887$$

$$u\left(\tfrac{2}{3}, \tfrac{1}{9}\right) \approx u_{2,1} = 0{,}2887 .$$

Vi kunne ha brukt vektornotasjon og (4'). Da får man

$$\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} u_{1,1} \\ u_{2,1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix} + \begin{pmatrix} u_{0,0} + u_{0,1} \\ u_{3,0} + u_{3,1} \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin\frac{\pi}{3} \\ \sin\left(\frac{2\pi}{3}\right) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

og man får det samme som (5)

TK