TMA4125 Matematikk 4N

Øving 1 - Laplacetransform I - LF

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Obligatoriske oppgaver

2 Matlab:

```
%kommando for aa slette alt i alle variabler
clear

%dette her er massevis av punkter paa t-aksen
t=[-pi:.1:pi];

%dette er funksjonen evaluert i ovennevnte punkter
f=cos(3*t).*cos(2*t);

%i matlab plotter man vektorer mot hverandre
plot(t,f)

%her er kommando for aa sette aksene
axis([-pi pi -1 1])

%til slutt en kommando for aa plotte t-aksen. 'hold on' sorger for at dette
%plottet kommer i samme figur som plottet over.
hold on
plot(t,zeros(length(t)),'red')
```

Python:

```
import numpy as np
import matplotlib.pyplot as plt

N=100

#t-aksen
t=np.linspace(-np.pi,np.pi,num=N)

#funksjonsverdiene
f=np.cos(2*t)*np.cos(3*t)

#lage plot
plt.plot(t,f)

#lage plot
plt.plot(t,np.zeros(N))

#korrekt utsnitt
plt.axis([-np.pi,np.pi,-1,1])

# navn paa aksene
plt.xlabel(r'$x$')
plt.ylabel(r'$y$')

#vise plot
plt.show()
```

$\boxed{3}$ **a)** Merk at

$$\sinh t \cos t = \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{2} \cos t = \frac{\mathrm{e}^t}{2} \cos t - \frac{\mathrm{e}^{-t}}{2} \cos t.$$

Siden

$$\mathscr{L}\{\cos t\} = \frac{s}{s^2+1},$$

kan vi bruke s-skft, og få

$$\begin{aligned} F(s) = \mathscr{L}\{f(t)\} &= \mathscr{L}\left\{\frac{\mathrm{e}^t}{2}\cos t - \frac{\mathrm{e}^{-t}}{2}\cos t\right\} = \frac{1}{2}\mathscr{L}\left\{\mathrm{e}^t \cos t\right\} - \frac{1}{2}\mathscr{L}\left\{\mathrm{e}^{-t}\cos t\right\} \\ &= \frac{1}{2}\frac{s-1}{(s-1)^2+1} - \frac{1}{2}\frac{s+1}{(s+1)^2+1}. \end{aligned}$$

b) Husk at

$$\cos^2 x = \frac{1+\cos 2x}{2}.$$

I tabellen finner vi

$$\mathscr{L}\{1\} = \frac{1}{s} \quad \text{and} \quad \mathscr{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$$

og lineariteten til laplacetransform gir

$$\begin{aligned} F(s) = \mathscr{L}\{f(t)\} &= \mathscr{L}\left\{\frac{1+\cos 4t}{2}\right\} = \frac{1}{2}\mathscr{L}\{1\} + \frac{1}{2}\mathscr{L}\{\cos 4t\} \\ &= \frac{1}{2s} + \frac{1}{2}\frac{s}{s^2+16} = \frac{1}{2s}\frac{2s^2+16}{s^2+16} = \frac{1}{s}\frac{s^2+8}{s^2+16}. \end{aligned}$$

Vi kan også skrive

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\cos^2 2t = -4\cos 2t \sin 2t = -2\sin 4t.$$

Derivasjonsformelen

$$\mathscr{L}\{f'\} = s\,\mathscr{L}\{f\} - f(0)$$

gir da

$$\begin{aligned} \mathscr{L}\{f'\} &= -2\,\mathscr{L}\{\sin 4t\} = -\frac{2\cdot 4}{s^2+4^2} \\ &= s\,\mathscr{L}\{f\} - f(0) = sF(s) - 1. \end{aligned}$$

Vi stokker om, og får

$$F(s) = \frac{1}{s}\left(1 - \frac{8}{s^2+4^2}\right) = \frac{1}{s}\frac{s^2+8}{s^2+16}.$$

$\boxed{4}$ **a)** Siden $s^2 - 2s - 3 = (s+1)(s-3)$ kan vi delbrøksoppspalte

$$\begin{aligned} \frac{4}{s^2-2s-3} &= \frac{4}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3} = \frac{(A+B)s - 3A + B}{(s+1)(s-3)} \\ &\Rightarrow \quad \left.\begin{aligned} A + B &= 0 \\ -3A + B &= 4 \end{aligned}\right\} \quad \Rightarrow \quad A = -1, \quad B = 1. \end{aligned}$$

Vi fortsetter med

$$
\begin{aligned}
f(t) = \mathscr{L}^{-1}\{F(s)\} &= \mathscr{L}^{-1}\left\{-\frac{1}{s+1} + \frac{1}{s-3}\right\} = -\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} \\
&= -\mathrm{e}^{-t} + \mathrm{e}^{3t}.
\end{aligned}
$$

Merk at vi også kan skrive

$$
\begin{aligned}
f(t) = \mathscr{L}^{-1}\{F(s)\} &= 4\,\mathscr{L}^{-1}\left\{\frac{1}{(s+1)(s-3)}\right\} = \frac{4}{-1-3}\left(\mathrm{e}^{-t} - \mathrm{e}^{3t}\right) \\
&= -\mathrm{e}^{-t} + \mathrm{e}^{3t}.
\end{aligned}
$$

c) Husk at

$$
G(s) = \frac{1}{s^2-1} \quad \Rightarrow \quad g(t) = \mathscr{L}^{-1}\{G(s)\} = \sinh t.
$$

Vi fortsetter med

$$
\begin{aligned}
h(t) = \mathscr{L}^{-1}\{H(s)\} &= \mathscr{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \int_0^t g(\tau)\,\mathrm{d}\tau = \int_0^t \sinh\tau\,\mathrm{d}\tau = [\cosh\tau]_0^t \\
&= \cosh t - 1,
\end{aligned}
$$

og videre

$$
\begin{aligned}
f(t) = \mathscr{L}^{-1}\{F(s)\} &= \mathscr{L}^{-1}\left\{\frac{1}{s}H(s)\right\} = \int_0^t h(\tau)\,\mathrm{d}\tau = \int_0^t \cosh t - 1\,\mathrm{d}\tau \\
&= [\sinh\tau - \tau]_0^t = \sinh t - t.
\end{aligned}
$$

Merk at, siden $s^4 - s^2 = s^2(s^2-1)$, kan vi også delbrøksoppspalte

$$
\frac{1}{s^4-s^2} = \frac{1}{s^2(s^2-1)} = \frac{A+Bs}{s^2} + \frac{C+Ds}{s^2-1} = \frac{(B+D)s^3 + (A+C)s^2 - Bs - A}{s^2(s^2-1)}
$$

$$
\Rightarrow \quad \left.\begin{aligned} B+D &= 0 \\ A+C &= 0 \\ -B &= 0 \\ -A &= 1 \end{aligned}\right\} \quad \Rightarrow \quad A = -1, \quad B = 0, \quad C = 1, \quad D = 0,
$$

og skrive

$$
\begin{aligned}
f(t) = \mathscr{L}^{-1}\{F(s)\} &= \mathscr{L}^{-1}\left\{-\frac{1}{s^2} + \frac{1}{s^2-1}\right\} = -\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\right\} \\
&= -t + \sinh t.
\end{aligned}
$$

$\boxed{5}$ TMA4130 H17 oppgave 3

Frivillige oppgaver

$\boxed{1}$ Et enkelt varabelskift ($u = ct$) gir

$$
G(s) = \mathscr{L}\{f(ct)\}(s) = \int_0^\infty f(ct)\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{c}\int_0^\infty f(u)\mathrm{e}^{-su/c}\,\mathrm{d}u = \frac{1}{c}\mathscr{L}\{f(t)\}\left(\frac{s}{c}\right) = \frac{1}{c}F\left(\frac{s}{c}\right).
$$

$\boxed{3}$ Vi beregner

$$\begin{aligned}\mathscr{L}\{f\} &= \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \sum_{n=0}^{\infty}\int_{np}^{np+p} f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \sum_{n=0}^{\infty}\int_0^p f(\tau+np)\mathrm{e}^{-s(\tau+np)}\,\mathrm{d}\tau \\ &= \sum_{n=0}^{\infty}\mathrm{e}^{-snp}\int_0^p f(\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau = \int_0^p f(t)\mathrm{e}^{-st}\,\mathrm{d}t\sum_{n=0}^{\infty}\left(\mathrm{e}^{-sp}\right)^n \\ &= \frac{1}{1-\mathrm{e}^{-ps}}\int_0^p f(t)\mathrm{e}^{-st}\,\mathrm{d}t\end{aligned}$$

der vi har brukt formelen for summen av en geometrisk rekke:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x|<1.$$

Merk at $|\mathrm{e}^{-ps}|<1$, siden $s,p>0$.

$\boxed{4}$ **b)** Let $Y(s)$ denote $\mathcal{L}(y)(s)$. We take the Laplace transform on both sides of the differential equation and use the formula for the Laplace transform of derivatives. This gives the subsidiary equation:

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + 2Y(s) = 6\mathcal{L}\left(e^{-t}\right)(s). \tag{1}$$

Using the given initial values $y'(0)=1, y(0)=0$, we obtain:

$$s^2Y(s) - 1 - 2sY(s) + 2Y(s) = \frac{6}{s+1}. \tag{2}$$

That is:

$$Y(s)\left(s^2-2s+2\right) = \frac{6}{s+1} + 1. \tag{3}$$

We can write $(s^2-2s+2) = (s-1)^2+1$ (transfer function). Thus we have:

$$Y(s)\left(s-1)^2+1\right) = \frac{6}{s+1} + 1. \tag{4}$$

That is:

$$Y(s) = \frac{6}{(s+1)((s-1)^2+1)} + \frac{1}{(s-1)^2+1}. \tag{5}$$

For the first term, we write:

$$\frac{6}{(s+1)((s-1)^2+1} = \frac{A}{s+1} + \frac{Bs+C}{(s-1)^2+1} \tag{6}$$

and try to determine the constants A,B,C. Expanding the right-hand side over a common denominator we are left with the equation:

$$\begin{aligned}6 &= A(s^2-2s+2) + Bs(s+1) + C(s+1) \\ &= s^2(A+B) + s(B-2A+C) + 2A + C.\end{aligned} \tag{7}$$

Comparing coefficients of powers of s on both sides gives then the following equations:

$$\begin{aligned} A + B &= 0 && (8)\\ B - 2A + C &= 0 && (9)\\ 2A + C &= 6. && (10) \end{aligned}$$

The first of these gives $A = -B$ which inserted in the second gives $-3A+C = 0$ or $C = 3A$. Then finally inserted in the last equation we get $5A = 6$ or $A = \frac{6}{5}$. Working backwards and plugging in then gives $B = -\frac{6}{5}$ and $C = \frac{18}{5}$. Thus we find:

$$\begin{aligned} Y(s) &= \frac{6/5}{s+1} + \frac{18/5}{(s-1)^2+1} - \frac{6s/5}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}\\ &= \frac{6/5}{s+1} + \frac{23/5}{(s-1)^2+1} - \frac{6s/5}{(s-1)^2+1}\\ &= \frac{6/5}{s+1} + \frac{17/5}{(s-1)^2+1} - \frac{(6/5)(s-1)}{(s-1)^2+1}, \qquad (11) \end{aligned}$$

from which we conclude that (use **f)** and **g)** from $\boxed{1}$, or use the s-shifting formula):

$$\begin{aligned} y(t) &= \frac{6}{5}e^{-t} + \frac{17}{5}e^t \sin(t) - \frac{6}{5}e^t \cos(t)\\ &= \frac{1}{5}\left(6e^{-t} + e^t\left(17\sin(t) - 6\cos(t)\right)\right). \qquad (12) \end{aligned}$$

$\boxed{5}$ **a)** We know that $\mathcal{L}\left(e^{At}\right)(s) = \frac{1}{s-A}$. Also, by definition, $\sinh(At) = \frac{e^{At}-e^{-At}}{2}$. Hence, by linearity of the Laplace transform, we get:

$$\begin{aligned} \mathcal{L}\left(\sinh(At)\right)(s) &= \mathcal{L}\left(\frac{e^{At}-e^{-At}}{2}\right)(s)\\ &= \frac{1}{2}\left(\mathcal{L}\left(e^{At}\right)(s) - \mathcal{L}\left(e^{-At}\right)(s)\right)\\ &= \frac{1}{2}\left(\frac{1}{s-A} - \frac{1}{s+A}\right)\\ &= \frac{A}{s^2-A^2}. \qquad (13) \end{aligned}$$

A slightly different approach using relations resulting from complex numbers is as follows. We know the Laplace transform of $\sin(At)$: $\mathcal{L}\left(\sin(At)\right)(s) = \frac{A}{s^2+A^2}$, and the we know the identity: $\sin(At) = \frac{e^{iAt}-e^{-iAt}}{2i}$. So it follows that $i\sin(Ait) = \frac{e^{-At}-e^{At}}{2} = -\sinh(At)$. Thus we conclude that:

$$\begin{aligned} \mathcal{L}\left(\sinh(At)\right)(s) &= -i\mathcal{L}\left(\sin(Ait)\right)(s)\\ &= -i\frac{Ai}{s^2+(Ai)^2}\\ &= \frac{A}{s^2-A^2}. \qquad (14) \end{aligned}$$

b) This is similar to **a)**, but we now use the definition of $\cosh(At)$, instead of $\sinh(At)$. That is, $\cosh(At) = \frac{e^{At}+e^{-At}}{2}$. Then again, by linearity of the Laplace transform, it follows that:

$$\begin{aligned}\mathcal{L}(\cosh(At))(s) &= \frac{1}{2}\left(\mathcal{L}\left(e^{At}\right)(s) + \mathcal{L}\left(e^{-At}\right)(s)\right)\\ &= \frac{1}{2}\left(\frac{1}{s-A} + \frac{1}{s+A}\right)\\ &= \frac{s}{s^2-A^2}. \end{aligned} \tag{15}$$

As in **a)**, there exists an approach using complex numbers here as well. We know the Laplace transform of the function $\cos(At)$: $\mathcal{L}(\cos(At))(s) = \frac{s}{s^2+A}$ and also $\cos(At) = \frac{e^{iAt}+e^{-iAt}}{2}$. From the latter it follows that $\cos(iAt) = \cosh(At)$ and hence:

$$\mathcal{L}(\cosh(At))(s) = \mathcal{L}(\cos(Ait))(s) = \frac{s}{s^2+(Ai)^2} = \frac{s}{s^2-A^2}. \tag{16}$$

c) Here we can use the definition of the Laplace transform directly. Because we have that $f(t) = 0$ for $t \in (0, \pi)$, and is equal to 1 elsewhere, we get:

$$\mathcal{L}(f)(s) = \int_\pi^\infty e^{-st}\, dt = \frac{e^{-\pi s}}{s}. \tag{17}$$

d) As in **c)**, we only integrate starting from $t > \pi$ in the definition of the Laplace transform of the function f. This gives:

$$\mathcal{L}(f)(s) = \int_\pi^\infty \cos(t)e^{-st}\, dt. \tag{18}$$

One can deal with this integral using partial integration (check!). An arguably quicker way is to use complex numbers. Note that $\cos(t)$ is the real part of $e^{it} = \cos(t) + i\sin(t)$ (Euler's identity) and hence we expect $\mathcal{L}(f)(s)$ to be the real part of the function $\int_\pi^\infty e^{it}e^{-st}\, dt = \int_\pi^\infty e^{t(i-s)}\, dt$. The latter integral computes as follows:

$$\int_\pi^\infty e^{t(i-s)}\, dt = \frac{e^{t(i-s)}}{i-s}\Big|_\pi^\infty = \frac{e^{\pi(i-s)}}{s-i} = \frac{e^{-\pi s}\cos(\pi)}{s-i} = \frac{\cos(\pi)e^{-\pi s}(s+i)}{s^2+1}. \tag{19}$$

The real part of the right-hand side of the last equality is $-\frac{se^{-\pi s}}{s^2+1}$; recall that $\cos(\pi) = -1$, and thus we conclude that:

$$\mathcal{L}(f)(s) = -\frac{se^{-\pi s}}{s^2+1}. \tag{20}$$

e) One way is to use the s-shift theorem. Let $F(s)$ denote the Laplace transform $\mathcal{L}(f)(s)$. Then by definition we get:

$$\mathcal{L}\left(e^{at}f(t)\right)(s) = \int_0^\infty e^{-(s-a)t}f(t)\, dt = F(s-a). \tag{21}$$

Let us put $f(t) = t^2$. Then we want $\mathcal{L}\left(e^t f(t)\right)(s) = F(s-1)$. We recall that $\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$ for $n = 0, 1, 2, \ldots$. Hence $F(s) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$ and hence:

$$\mathcal{L}\left(t^2e^t\right)(s) = F(s-1) = \frac{2}{(s-1)^3}. \tag{22}$$

Another approach is to do a partial integration and deduce the transform of t^2e^t directly from the definition (check!).

f) One possibility is to do as in part **d)** and use s-shifting. Let $f(t) = \cos(t)$. Then $\mathcal{L}\left(e^t\cos(t)\right)(s) = F(s-1) = \frac{s-1}{(s-1)^2+1}$. A second approach is to use partial integration together with the definition directly (check!). A third approach is based on using complex numbers. We note that $\cos(t)$ is the real part of e^{it}. Thus we expect the Laplace transform of $e^t\cos(t)$ to be the real part of the Laplace transform of $e^te^{it} = e^{t(i+1)}$. This is $\frac{1}{s-(i+1)} = \frac{1}{s-i-1} = \frac{s+i-1}{(s-1)^2+1}$. The real part of this is then $\frac{s-1}{(s-1)^2+1}$; the same answer as before, when we used s-shifting.

g) This is almost the same as **f)**. We can use s-shifting from which we immediately get, letting $F(s) = \mathcal{L}(f)(s)$ with $f(t) = \sin(t)$, and thus with $F(s) = \frac{1}{s^2+1}$:

$$\mathcal{L}\left(e^t\sin(t)\right)(s) = \mathcal{L}\left(e^tf(t)\right)(s) = F(s-1) = \frac{1}{(s-1)^2+1}. \tag{23}$$

Or we can use complex numbers and realize $\mathcal{L}\left(e^t\sin(t)\right)(s)$ as the imaginary part of the integral $\int_0^\infty e^{t(i+1-s)}\,dt = \frac{1}{s-i-1} = \frac{s+i-1}{(s-1)^2+1}$. The imaginary part of the latter is $\frac{1}{(s-1)^2+1}$, which is the same answer as before when we used s-shifting.