TMA4125 Matematikk 4N

Øving 2 - Laplacetransform II - LF

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Obligatoriske oppgaver

$\boxed{1}$ See the Lecture notes

$\boxed{2}$ **Matlab:**

```
clear
%se oving_1.m hvis du lurer paa noe her
t=[-pi:.01:pi];
f=heaviside(t);
plot(t,f)
axis([-pi pi -1 2])
```

Python:

```
import numpy as np
import matplotlib.pyplot as plt

N=1000

#t-aksen
t=np.linspace(-np.pi,np.pi,num=N)

# det andre argumentet er funksjonsverdien for x1=0.
f = np.heaviside(t,0)

#lage plot av funksjonen
plt.plot(t,f)

#korrekt utsnitt
plt.axis([-np.pi,np.pi,-1,2])

# navn paa aksene
plt.xlabel(r'$x$')
plt.ylabel(r'$y$')

#vise plot
plt.show()
```

$\boxed{3}$ a) Taking the Laplace transform of

$$y''(t) + 4y'(t) + 5y(t) = \delta(t-1)$$

yields

$$s^2Y(s) - sy(0) - y'(0) + 4\left(sY(s) - y(0)\right) + 5Y(s) = \mathrm{e}^{-s}.$$

With the initial conditions $y(0) = 0$ and $y'(0) = 3$ we have

$$s^2Y(s) - 3 + 4sY(s) + 5Y(s) = \mathrm{e}^{-s}$$

such that

$$Y(s) = \frac{\mathrm{e}^{-s} + 3}{s^2 + 4s + 5} = \frac{3}{(s-(-2))^2 + 1^2} + \frac{\mathrm{e}^{-1\cdot s}}{(s-(-2))^2 + 1^2}$$

Taking the inverse Laplace transform (using s-shifting , t-shifting and a standard table of Laplace transforms (link)

$$y(t) = 3\mathrm{e}^{-2t}\sin t + \mathrm{e}^{-2(t-1)}\sin(t-1)u(t-1)$$

b) Taking the Laplace transform of

$$\begin{aligned} y''(t) + 5y'(t) + 6y(t) &= \delta\left(t - \frac{\pi}{2}\right) + u(t-\pi)\cos t \\ &= \delta\left(t - \frac{\pi}{2}\right) - u(t-\pi)\cos(t-\pi) \end{aligned}$$

yields

$$s^2Y(s) - sy(0) - y'(0) + 5\left(sY(s) - y(0)\right) + 6Y(s) = \mathrm{e}^{-\pi s/2} - \frac{s\mathrm{e}^{-\pi s}}{s^2+1}.$$

With the initial conditions $y(0) = 0$ and $y'(0) = 0$ we have

$$s^2Y(s) + 5sY(s) + 6Y(s) = \mathrm{e}^{-\pi s/2} - \frac{s\mathrm{e}^{-\pi s}}{s^2+1}$$

such that

$$Y(s) = \frac{\mathrm{e}^{-\pi s/2}}{s^2+5s+6} - \frac{s\mathrm{e}^{-\pi s}}{(s^2+1)(s^2+5s+6)}.$$

Using partial fraction expansion we get $(s^2 + 5s + 6 = (s + 2)(s + 3))$

$$\begin{aligned} \frac{1}{s^2+5s+6} &= \frac{A}{s+2} + \frac{B}{s+3} = \frac{(A+B)s + 3A + 2B}{s^2+5s+6} \\ &\Rightarrow \begin{cases} A + B = 0 \\ 3A + 2B = 1 \end{cases} \\ &\Rightarrow \quad A = 1, \quad B = -1 \end{aligned}$$

and

$$\begin{aligned} &\frac{s}{(s^2+1)(s^2+5s+6)} = \frac{A+Bs}{s^2+1} + \frac{C}{s+2} + \frac{D}{s+3} = \frac{A+Bs}{s^2+1} + \frac{(C+D)s + 3C + 2D}{(s^2+5s+6)} \\ &= \frac{(B+C+D)s^3 + (A+5B+3C+2D)s^2 + (5A+6B+C+D)s + 6A + 3C + 2D}{(s^2+1)(s^2+5s+6)} \\ &\Rightarrow \begin{cases} B + C + D = 0 \\ A + 5B + 3C + 2D = 0 \\ 5A + 6B + C + D = 1 \\ 6A + 3C + 2D = 0 \end{cases} \\ &\Rightarrow \quad A = \frac{1}{10}, \quad B = \frac{1}{10}, \quad C = -\frac{2}{5}, \quad D = \frac{3}{10}. \end{aligned}$$

Hence,

$$\begin{aligned} Y(s) &= \left(\frac{1}{s+2} - \frac{1}{s+3}\right)\mathrm{e}^{-\pi s/2} - \left(\frac{1}{10}\frac{1+s}{s^2+1} - \frac{2}{5}\frac{1}{s+2} + \frac{3}{10}\frac{1}{s+3}\right)\mathrm{e}^{-\pi s} \\ &= \left(\frac{1}{s+2} - \frac{1}{s+3}\right)\mathrm{e}^{-s\pi/2} - \frac{1}{10}\left(\frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{4}{s+2} + \frac{3}{s+3}\right)\mathrm{e}^{-\pi s} \end{aligned}$$

Taking the inverse Laplace transform (using t-shifting and a table of Laplace transforms)

$$\begin{aligned}
y(t) &= \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}\right) u\left(t - \frac{\pi}{2}\right) \\
&\quad - \frac{1}{10}\left(\sin(t-\pi) + \cos(t-\pi) - 4e^{-2(t-\pi)} + 3e^{-3(t-\pi)}\right) u(t-\pi) \\
&= \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}\right) u\left(t - \frac{\pi}{2}\right) \\
&\quad + \frac{1}{10}\left(\sin t + \cos t + 4e^{-2(t-\pi)} - 3e^{-3(t-\pi)}\right) u(t-\pi)
\end{aligned}$$

c) We have

$$\mathscr{L}\{ty'(t)\} = -\frac{d}{ds}\left(sY(s) - y(0)\right) = -Y(s) - sY'(s)$$

and (using $y(0) = 1$)

$$\mathscr{L}\{ty''(t)\} = -\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right) = -2sY(s) - s^2Y'(s) + 1.$$

Taking the Laplace transform of the ODE therefore yields

$$\begin{aligned}
&-2sY(s) - s^2Y'(s) + 1 + Y(s) + sY'(s) + Y(s) = \frac{1}{s} \\
&\Rightarrow Y'(s) + \frac{2}{s}Y(s) = \frac{1}{s^2}.
\end{aligned}$$

From earlier courses we know that we can solve this *first* order ODE using integrating factor (which is s^2 in this case). Multiplication by the integrating factor yields

$$\begin{aligned}
s^2Y'(s) + 2sY(s) = 1 \quad \Rightarrow \quad &\frac{d}{ds}\left(s^2Y(s)\right) = 1 \quad \Rightarrow \quad s^2Y(s) = s + C \\
\Rightarrow\ &Y(s) = \frac{1}{s} + \frac{C}{s^2}
\end{aligned}$$

for some constant C. Taking the inverse transform

$$y(t) = 1 + Ct.$$

Since $y'(0) = 2$ we have $C = 2$. The solution of the initial value problem is therefore

$$y(t) = 1 + 2t.$$

d) Note that we can write the integral equation in the form

$$y(t) - (y * g)(t) = 2 - \frac{1}{2}t^2$$

with $g(t) = t$. Using the convolution theorem and the infamous table we get

$$Y(s) - \frac{Y(s)}{s^2} = \frac{2}{s} - \frac{1}{s^3} \quad \Rightarrow \quad Y(s) = \frac{2s^2 - 1}{s(s^2 - 1)},$$

and by partial fraction expansion we get

$$\begin{aligned}\frac{2s^2-1}{s(s^2-1)} &= \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+1} = \frac{A}{s}+\frac{(B+C)s+B-C}{s^2-1}\\ &= \frac{(A+B+C)s^2+(B-C)s-A}{s(s^2-1)}\\ &\Rightarrow \quad \begin{cases} A+B+C=2\\ B-C=0\\ -A=-1\end{cases}\\ &\Rightarrow \quad A=1, \quad B=\frac{1}{2}, \quad C=\frac{1}{2}.\end{aligned}$$

Hence,

$$Y(s)=\frac{1}{s}+\frac{1}{2}\frac{1}{s-1}+\frac{1}{2}\frac{1}{s+1}.$$

Taking the inverse Laplace transform (look here) yields

$$y(t)=1+\frac{1}{2}\mathrm{e}^{t}+\frac{1}{2}\mathrm{e}^{-t}=1+\cosh t.$$

Anbefalte oppgaver

$\boxed{1}$ See the Lecture notes

$\boxed{2}$ **a)** We write $\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}$. Hence:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right)(t)=\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)(t)-\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)(t)=t-\sin(t). \tag{1}$$

b) We write $\frac{s}{s^2+2s+1}=\frac{s}{(s+1)^2}=\frac{s+1-1}{(s+1)^2}=\frac{1}{s+1}-\frac{1}{(s+1)^2}$. Hence, using also s-shifting:

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+1}\right)(t)=\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)(t)-\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)(t)=e^{-t}-e^{-t}t. \tag{2}$$

That is $\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+1}\right)(t)=e^{-t}(1-t)$.

c) We write: $\frac{2s}{(s^2+1)^2}=\frac{2s}{s^2+1}\cdot\frac{1}{s^2+1}:=F(s)G(s)$, where we recognize that $F(s)$ and $G(s)$ are Laplace transforms of some known functions: $F(s)=2\mathcal{L}(\cos(t))(s)$ and $G(s)=\mathcal{L}(\sin(t))(s)$. Thus by the convolution formula:

$$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right)(t)=(2\cos(t)*\sin(t)). \tag{3}$$

By definition of the convolution, we get:

$$\begin{aligned}
2\cos(t) * \sin(t) &= \int_0^t 2\cos(\tau)\sin(t-\tau)\, d\tau \\
&= 2\int_0^t \cos(\tau)\left(\sin(t)\cos(\tau) - \sin(\tau)\cos(t)\right)\, d\tau \\
&= 2\sin(t)\int_0^t \cos^2(\tau)\, d\tau - 2\cos(t)\int_0^t \cos(\tau)\sin(\tau)\, d\tau \\
&= 2\sin(t)\int_0^t \frac{\cos(2\tau)+1}{2}\, d\tau - 2\cos(t)\int_0^t \frac{1}{2}\sin(2\tau)\, d\tau \\
&= 2\sin(t)\left(\frac{\sin(2\tau)}{4} + \frac{\tau}{2}\right)\Big|_0^t - 2\cos(t)\left(-\frac{\cos(2\tau)}{4}\right)\Big|_0^t \\
&= \frac{1}{2}\sin(t)\sin(2t) + t\sin(t) + \frac{1}{2}\cos(t)\cos(2t) - \frac{1}{2}\cos(t). \qquad (4)
\end{aligned}$$

Now, we have:

$$\frac{1}{2}\sin(t)\sin(2t) = \sin(t)^2\cos(t) \qquad (5)$$

and

$$\frac{1}{2}\cos(t)\cos(2t) - \frac{1}{2}\cos(t) = \cos(t)\left(\frac{\cos(2t)-1}{2}\right) = -\cos(t)\sin^2(t) \qquad (6)$$

and hence it follows that only the term $t\sin t$ is left. That is:

$$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right)(t) = t\sin t. \qquad (7)$$

d) We write $(s-3)^{-5} = \frac{1}{(s-3)^5}$. Now recall that, if $h_m = t^m$, then $\mathcal{L}(h_m)(s) = \frac{m!}{s^{m+1}}$ when $m \in \mathbb{N} \cup \{0\}$. We take $m = 4$ so that $m+1 = 5$. Then we see that $\mathcal{L}(h_4)(s) = \frac{4!}{s^5} = \frac{24}{s^5}$. Considering this and s-shifting, we conclude that:

$$\mathcal{L}^{-1}\left((s-3)^{-5}\right)(t) = \frac{1}{24}e^{3t}t^4. \qquad (8)$$

$\boxed{3}$ **a)** We first recall the t-shifting formula. Let θ_a be the function defined to be given by $\theta_a(t) = u(t-a)$. That is the unit-step function at a. We consider $\mathcal{L}(\theta_a f)(s)$ for some general function f whose Laplace transform exists. Then by definition:

$$\begin{aligned}
\mathcal{L}(\theta_a f)(s) &= \int_0^\infty e^{-st}\theta_a(t)f(t)\, dt \\
&= \int_0^\infty e^{-st}u(t-a)f(t)\, dt \\
&= \int_a^\infty e^{-st}f(t)\, dt. \qquad (9)
\end{aligned}$$

The last equality follows because $u(t-a) = 0$ for $t < a$ and is equal to 1 when $t > a$. This is by definition. Now, let us change variables. We define $x := t-a$. Then x runs from $x = 0$ to $x = \infty$. Moreover $dx = dt$ and $t = x + a$. Substituting this, we find:

$$\mathcal{L}(\theta_a f)(s) = \int_0^\infty e^{-s(x+a)}f(x+a)\, dx = e^{-sa}\int_0^\infty e^{-sx}f(x+a)\, dx. \qquad (10)$$

Now, let us define f_{-a} to be the function such that $f_{-a}(x) = f(x+a)$. The integral on the right-hand side is then by definition $\mathcal{L}(f_{-a})(s)$. Hence we conclude that:

$$\mathcal{L}(\theta_a f)(s) = e^{-sa}\mathcal{L}(f_{-a})(s). \tag{11}$$

A perhaps more familiar way of writing this, so that the Laplace transform of f appears on the right-hand side is then:

$$\mathcal{L}(\theta_a f_a)(s) = e^{-sa}\mathcal{L}(f)(s). \tag{12}$$

Using the notation in the book, this is:

$$\mathcal{L}(u(t-a)f(t-a))(s) = e^{-sa}\mathcal{L}(f)(s). \tag{13}$$

We now consider the Laplace transform of f from the exercise. That is, of f, where $f(t) = (\theta_0(t) - \theta_\pi(t))\cos(t)$. According to the formula we derived above, we have:

$$\begin{aligned}\mathcal{L}(\theta_0(t)\cos(t))(s) &= e^{-0s}\mathcal{L}(\cos(t+0)(s) \\ &= 1 \cdot \mathcal{L}(\cos(t))(s) \\ &= \frac{s}{s^2+1}.\end{aligned} \tag{14}$$

Similarly, from the formula above, we have with $a = \pi$:

$$\begin{aligned}\mathcal{L}(\theta_\pi(t)\cos(t))(s) &= e^{-s\pi}\mathcal{L}(\cos(t+\pi))(s) \\ &= e^{-\pi s}\mathcal{L}(-\cos(t))(s) \\ &= -\frac{se^{-\pi s}}{s^2+1}.\end{aligned} \tag{15}$$

Thus we finally have:

$$\mathcal{L}(f)(s) = \frac{s}{s^2+1} + \frac{se^{-\pi s}}{s^2+1} = \frac{s(1+e^{-\pi s})}{s^2+1}. \tag{16}$$

To draw the graph of f, we consider how $u(t) - u(t-\pi)$ behaves like for different values of t. Note that for $t < 0$ both terms $u(t)$ and $u(t-\pi)$ are zero, so the whole expression is 0. When $0 < t < \pi$, $u(t) = 1$ while $u(t-\pi) = 0$. Hence the whole expression is equal to 1. Finally for $t > \pi$, $u(t) = 1$ and $u(t-\pi) = 1$. So the whole expression is again zero. Thus the only place it is non-zero, in which case it is equal to 1, is when $t \in (0, \pi)$. Thus the graph of f is the graph of $\cos(t)$ for $t \in (0, \pi)$ and zero elsewhere.

b) We use the t-shifting formula. Let $h_2(t) = t^2$. We get:

$$\mathcal{L}(\theta_a h_2)(s) = e^{-sa}\mathcal{L}((h_2)_{-a})(s). \tag{17}$$

We have $(h_2)_{-a}(t) = (t+a)^2 = t^2 + 2ta + a^2$. Hence:

$$\begin{aligned}\mathcal{L}((h_2)_{-a})(s) &= \mathcal{L}\left(t^2 + 2ta + a^2\right)(s) \\ &= \frac{2}{s^3} + \frac{2a}{s^2} + \frac{a^2}{s}\end{aligned} \tag{18}$$

and hence:

$$\mathcal{L}(\theta_a h_2)(s) = e^{-as}\left(\frac{2}{s^3} + \frac{2a}{s^2} + \frac{a^2}{s}\right). \tag{19}$$

As for the graph of $\theta_a h_2$, we use the definition of θ_a and h_2. It follows that $(\theta_a h_2)(t) = 0$ for $t < a$ and is equal to $h_2(t) = t^2$ for $t > a$.

c) Perhaps the simplest way to solve this exercise is to look at the graph of f (given below with $a = \pi$) and see that it alternates between -1 and 1 in intervals of length a. Thus we can split the integral in the Laplace transform to get

$$\begin{aligned}
\mathcal{L}(f)(s) &= \sum_{i=0}^{\infty}(-1)^i \int_{ai}^{a(i+1)} e^{-st}\,dt = \sum_{i=0}^{\infty}(-1)^i \Big[\frac{-1}{s}e^{-st}\Big]\Big|_{ai}^{a(i+1)} \\
&= \sum_{i=0}^{\infty}(-1)^i \frac{-1}{s}(e^{-sa(i+1)} - e^{-sai}) \\
&= \sum_{i=0}^{\infty}(-1)^i \frac{1}{s}e^{-sai}(1 - e^{-sa}) \\
&= \sum_{i=0}^{\infty}\frac{1 - e^{-sa}}{s}(-e^{-sa})^i. \qquad (20)
\end{aligned}$$

Now, since both s and a are positive, $|-e^{-sa}| < 1$, so we can use the formula for a geometric series to obtain

$$\begin{aligned}
\frac{1 - e^{-sa}}{s}\sum_{i=0}^{\infty}(-e^{-sa})^i &= \frac{1 - e^{-sa}}{s}\frac{1}{1 - (-e^{-sa})} \\
&= \frac{1 - e^{-sa}}{s(1 + e^{-sa})} \\
&= \frac{1}{s}\tanh(sa/2) \qquad (21)
\end{aligned}$$

Note: When we solve this exercise we assume that we can change the order of integration and summation. In general, for infinite sums, this cannot be done. In our case this is possible, since

$$\begin{aligned}
\int_0^{\infty} e^{-st}\sum_{i=1}^{\infty}(-1)^i u(t - ia)\,dt &= \int_0^{\infty} e^{-st}\Big(\sum_{i=1}^{N}(-1)^i u(t - ia) \\
&\quad + \sum_{i=N}^{\infty}(-1)^i u(t - ia)\Big)\,dt \\
&= \sum_{i=1}^{N}\int_0^{\infty} e^{-st}(-1)^i u(t - ia)\,dt \\
&\quad + \int_0^{\infty}\sum_{i=N}^{\infty}(-1)^i e^{-st}u(t - ia)\Big)\,dt \\
&= \sum_{i=1}^{N}\int_0^{\infty} e^{-st}(-1)^i u(t - ia)\,dt \\
&\quad + \int_{Na}^{\infty}\sum_{i=N}^{\infty}(-1)^i e^{-st}u(t - ia)\Big)\,dt. \qquad (22)
\end{aligned}$$

As $|(-1)^i u(t-ia)|$ is bounded by 1, we get the bound

$$
\begin{aligned}
|\int_{Na}^{\infty}\sum_{i=N}^{\infty}(-1)^i e^{-st}u(t-ia)\,dt| &\leq \int_{Na}^{\infty}\sum_{i=N}^{\infty}|(-1)^i e^{-st}u(t-ia)|\,dt \\
&\leq \int_{Na}^{\infty}\sum_{i=N}^{\infty}|e^{-st}|\,dt,
\end{aligned} \tag{23}
$$

which converges to zero when $N\to\infty$ since e^{-st} becomes very small.

$\boxed{4}$ **a)** We take the Laplace transform on both sides and use the initial conditions

$$
s^2Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0} + Y(s) = \frac{e^{-\pi s}}{s}.
$$

We collect the terms (using the flytte-bytte rules)

$$
Y(s) = \underbrace{\frac{1}{s(s^2+1)}}_{=:F(s)} e^{-\pi\cdot s} = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-\pi\cdot s}
$$

We note that the inverse Laplace transform of F is $f(t) = 1-\cos(t)$, and using t-shifting we get

$$
y(t) = (1-\cos(t-\pi))u(t-\pi).
$$

b) We take the Laplace transform on both sides and use that $\mathcal{L}(f*g)(s) = \mathcal{L}(f)(s)\mathcal{L}(g)(s)$. We set $\mathcal{L}(y)(s) := Y(s)$ and recall $\mathcal{L}(h_1)(s) = \frac{1}{s^2}$ where $h_1(t) = t$. Then we get after taking Laplace transform on both sides of the given equation:

$$
Y(s) - Y(s)\frac{1}{s^2} = \frac{1}{s^2}. \tag{24}
$$

That is:

$$
Y(s)\left(1-\frac{1}{s^2}\right) = Y(s)\left(\frac{s^2-1}{s^2}\right) = \frac{1}{s^2}. \tag{25}
$$

That is, for $s > |1|$, we obtain:

$$
Y(s) = \frac{1}{s^2-1}. \tag{26}
$$

Hence $y(t) = \sinh(t)$.

$\boxed{5}$ The RLC-circuit is governed by the following integro-differential equation

$$
Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau = v(t) \tag{27}
$$

where (in our case)

$$
\begin{aligned}
v(t) &= \begin{cases} -34\mathrm{e}^{-t} & 0<t<4 \\ 0 & \text{otherwise} \end{cases} \\
&= 34\mathrm{e}^{-t}(1-u(t-4)) = 34\mathrm{e}^{-t} - 34\mathrm{e}^{-4}\mathrm{e}^{-(t-4)}u(t-4)).
\end{aligned}
$$

The Laplace transform of this expression is given by (using t-shifting and a standard table of Laplace transform)

$$V(s) = \mathscr{L}\{v(t)\} = \frac{34}{s+1} - \frac{34\mathrm{e}^{-4}\mathrm{e}^{4s}}{s+1} = \frac{34\left(1-\mathrm{e}^{-4s-4}\right)}{s+1}.$$

Equating this result with the Laplace transform of the left hand side of (27), we obtain (using Laplace transform of derivatives and Laplace transform of integral)

$$\begin{aligned}\frac{34\left(1-\mathrm{e}^{-4s-4}\right)}{s+1} &= \mathscr{L}\left\{Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau\right\}\\ &= L\left(sI(s) - i(0)\right) + RI(s) + \frac{1}{C}\frac{I(s)}{s}\\ &= sI(s) + 4I(s) + 20\frac{I(s)}{s} = \left(s + 4 + \frac{20}{s}\right)I(s).\end{aligned}$$

Thus,

$$I(s) = \frac{34s\left(1-\mathrm{e}^{-4s-4}\right)}{(s+1)(s^2+4s+20)}.$$

Using partial fraction expansion we get

$$\begin{aligned}\frac{34s}{(s+1)(s^2+4s+20)} &= \frac{A}{s+1} + \frac{B+Ds}{s^2+4s+20}\\ &= \frac{(A+D)s^2 + (4A+B+D)s + 20A + B}{(s+1)(s^2+4s+20)}\\ &\Rightarrow \begin{cases} A + D = 0\\ 4A + B + D = 34\\ 20A + B = 0\end{cases}\\ &\Rightarrow \quad A = -2, \quad B = 40, \quad D = 2.\end{aligned}$$

Hence,

$$\begin{aligned}I(s) &= \left(-\frac{2}{s+1} + \frac{40+2s}{s^2+4s+20}\right)\left(1-\mathrm{e}^{-4s-4}\right)\\ &= \left(-\frac{2}{s-(-1)} + \frac{2(s-(-2))}{(s-(-2))^2+4^2} + 9\cdot\frac{4}{(s-(-2))^2+4^2}\right)\left(1-\mathrm{e}^{-4}\mathrm{e}^{-4s}\right).\end{aligned}$$

The inverse Laplace transform of this functions yields the final result (using s-shifting, t-shifting and standard table of Laplace transforms)

$$\begin{aligned}i(t) &= -2\mathrm{e}^{-1\cdot t} + 2\mathrm{e}^{-2t}\cos 4t + 9\mathrm{e}^{-2t}\sin 4t\\ &\quad - u(t-4)\mathrm{e}^{-4}\left(-2\mathrm{e}^{-1\cdot(t-4)} + 2\mathrm{e}^{-2(t-4)}\cos[4(t-4)] + 9\mathrm{e}^{-2(t-4)}\sin[4(t-4)]\right)\\ &= -2\mathrm{e}^{-t} + \mathrm{e}^{-2t}\left(2\cos 4t + 9\sin 4t\right)\\ &\quad - u(t-4)\left(-2\mathrm{e}^{-t} + \mathrm{e}^{-2(t-2)}\left(2\cos[4(t-4)] + 9\sin[4(t-4)]\right)\right).\end{aligned}$$

$\boxed{\text{Nøtt}}$ **a)** We first recall the definition of the Gamma function Γ (it is also stated in the exercise):

$$\Gamma(x+1) = \int_0^\infty z^x e^{-z}\,dz. \tag{28}$$

Let h_n be the function such that $h_n(t) = t^n$. Then by definition of the Laplace transform of h_n, we have:

$$\mathcal{L}(h_n)(s) = \int_0^\infty t^n e^{-st}\, dt. \tag{29}$$

We use a change of variables. Set $x := st$. Then, for $s > 0$, $t = \frac{x}{s}$ and $dt = \frac{1}{s}dx$. Also x runs from 0 to ∞. Substituting this, we get that:

$$\mathcal{L}(h_n)(s) = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\frac{1}{s}\, dx = \frac{1}{s^{n+1}}\int_0^\infty x^n e^{-x}\, dx = \frac{\Gamma(n+1)}{s^{n+1}}. \tag{30}$$

b) We first show $\Gamma(x+1) = x\Gamma(x)$. By definition we have:

$$\Gamma(x+1) = \int_0^\infty y^x e^{-y}\, dy. \tag{31}$$

We use integration by parts, which then yields:

$$\Gamma(x+1) = -y^x e^{-y}|_0^\infty + x\int_0^\infty y^{x-1}e^{-y}\, dy = x\Gamma(x)$$

where the last equality follows from the definition of $\Gamma(x)$ and the fact that $\lim_{y\to\infty} y^x e^{-y} = 0$. **Observation:** note that $\Gamma(1) = \int_0^\infty e^{-y}\, dy = 1$ and hence in the case $n \in \mathbb{N}$, it follows by this formula and by induction that $\Gamma(n+1) = n!$.

In order to calculate $\Gamma\left(\frac{2k+1}{2}\right)$ note that $\Gamma\left(\frac{2k+1}{2}\right) = \Gamma\left(k+\frac{1}{2}\right)$. From this and the formula we just proved ($\Gamma(x+1) = x\Gamma(x)$), it follows by induction that $\Gamma(k+1/2) = \Gamma((k-1)+1/2)+1) = \Gamma(1/2)\prod_{j=1}^k (j-1+1/2)$, when $k \geq 1$. (Here $\prod_{j=1}^k g_j$ for some quantities $g_1, \ldots, g_k$, simply means $g_1 g_2 \cdots g_k$; that is, the product of all these quantities.)

$$\begin{aligned}
\Gamma(3/2) &= \Gamma(1+1/2) = \frac{1}{2}\Gamma(1/2)\\
\Gamma(5/2) &= \Gamma(2+1/2) = \Gamma(1+3/2) = \frac{3}{2}\frac{1}{2}\Gamma(1/2)\\
\Gamma(7/2) &= \Gamma(3+1/2) = \Gamma(1+5/2) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(1/2)\\
\Gamma(9/2) &= \Gamma(4+1/2) = \Gamma(1+7/2) = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(1/2)
\end{aligned}$$

For the case $k = 0$, this reduces immediately to $\Gamma(1/2)$. Hence we will need to know how to calculate $\Gamma(1/2)$ in order for this to be useful. We do this in **d)** below.

c) By definition, we have:

$$\Gamma(1/2) = \int_0^\infty x^{-1/2}e^{-x}\, dx. \tag{32}$$

We use a change of variables. Set $u^2 := x$. Then u runs from 0 to ∞ and $dx = 2u\, du$. Substituting this, we therefore have:

$$\Gamma(1/2) = 2\int_0^\infty e^{-u^2}\, du. \tag{33}$$

d) We consider the integral $\int_0^\infty e^{-x^2}\, dx$. This is the classical *Gauss integral.* It has important applications in many areas, for instance in statistics where it appears in relation with the normal distribution. Denote the integral by I. Then we have $I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$. We now change coordinates to polar coordinates, setting $x^2 + y^2 = r$ and $dx\, dy = r\, dr\, d\theta$. Note that the integration area is over the whole first quadrant in $\mathbb{R}^2$. That is $\theta \in (0, \pi/2)$ and $r \in (0, \infty)$. Thus we find:

$$I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\, dr\, d\theta = \frac{\pi}{2} \int_0^\infty \frac{1}{2} e^{-u}\, du = \frac{\pi}{4} \tag{34}$$

and hence $I = \frac{\sqrt{\pi}}{2}$. In the second equality we have used the substitution $u := r^2$ giving $du = 2\, dr$.

e) We first note that by combining **c)** and **d)** we have $\Gamma(1/2) = \sqrt{\pi}$.

We expand the exponential, using the identity:

$$e^z = \sum_{n=0}^\infty \frac{z^n}{n!}. \tag{35}$$

Taking $z = -p^2$ we thus get:

$$e^{-p^2} = \sum_{n=0}^\infty \frac{(-1)^n p^{2n}}{n!}. \tag{36}$$

This is a power series with infinite radius of convergence (it converges for all p), and so from calculus we know that we can integrate term-wise over all of $\mathbb{R}$ (for those familiar with the term: we can do this because the series *converges uniformly* as we know from the theory of power series; c.f Abel's lemma). Hence integrating term-wise we find:

$$\begin{aligned}
\int_0^{\sqrt{t}} e^{-p^2}\, dp &= \int_0^{\sqrt{t}} \sum_{n=0}^\infty \frac{(-1)^n p^{2n}}{n!}\, dp \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^{\sqrt{t}} p^{2n}\, dp \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!(2n+1)} t^{n+1/2} \\
&= t^{1/2} - \frac{t^{3/2}}{3} + \frac{1}{5}\frac{t^{5/2}}{2!} - \frac{1}{7}\frac{t^{7/2}}{3!} + \cdots
\end{aligned} \tag{37}$$

Now we want to take the Laplace transform of the above series, and also here move the integral inside the sum:

$$\mathcal{L}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}t^{n+1/2}\right)(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\mathcal{L}\left(t^{n+1/2}\right)(s). \tag{38}$$

From **a)** we know the Laplace transform of $t^{n+1/2}$, i.e., $\mathcal{L}(t^{n+1/2})(s)=\frac{\Gamma(n+1+1/2)}{s^{n+1+1/2}}$, $s>0$, so we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\mathcal{L}\left(t^{n+1/2}\right)(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\frac{\Gamma(n+1+1/2)}{s^{n+1+1/2}}. \tag{39}$$

We now use **b)** with $x=n+1/2$, so

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\frac{\Gamma(n+1/2+1)}{s^{n+1/2+1}}&=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\frac{(n+1/2)\Gamma(n+1/2)}{s^{n+3/2}}. \\ &=\frac{\Gamma(3/2)}{s^{3/2}}-\frac{\Gamma(5/2)}{3s^{5/2}}+\frac{1}{5}\frac{\Gamma(7/2)}{2!s^{7/2}}+\frac{1}{7}\frac{\Gamma(9/2)}{3!s^{9/2}}+\cdots\end{aligned} \tag{40}$$

At last we use the formula for $\Gamma(k+\frac{1}{2})$ that we found in **b)** and what we noted in the beginning: that $\Gamma(1/2)=\sqrt{\pi}$, to get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!2}\frac{\Gamma(\frac{2n+1}{2})}{s^{n+3/2}}&=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!2}\frac{\Gamma(1/2)\prod_{j=1}^{n}(j-1/2)}{s^{n+3/2}} \\ &=\frac{\sqrt{\pi}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\prod_{j=1}^{n}(j-1/2)}{s^{n+3/2}} \\ &=\frac{\sqrt{\pi}}{2}\Big(\frac{1}{s^{3/2}}-\frac{1}{2}\frac{1}{s^{5/2}}+\frac{1}{2}\frac{3}{4}\frac{1}{s^{7/2}}-\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{s^{9/2}}+\cdots\Big) \\ &=\frac{\sqrt{\pi}}{2}\frac{1}{s^{3/2}}\Big(1-\frac{1}{2}\frac{1}{s}+\frac{1}{2}\frac{3}{4}\frac{1}{s^2}-\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{s^3}+\cdots\Big)\end{aligned} \tag{41, 42}$$

We introduce now the generalisation of what is known as binomial formula, i.e.,

$$(x+y)^r=\sum_{k=0}^{\infty}\binom{r}{k}x^{r-k}y^k,$$

which holds for $|x|>|y|$ and r is a real. The so-called Pochhammer symbol $\binom{r}{k}$ is defined by

$$\binom{r}{k}:=\frac{r(r-1)\cdots(r-k+1)}{k!}.$$

This then tells us that

$$(x+y)^r=x^r+rx^{r-1}y+\frac{r(r-1)}{2!}x^{r-2}y^2+\frac{r(r-1)(r-2)}{3!}x^{r-3}y^3+\cdots.$$

Hence, with $x = 1$, $y = 1/s$, $s > 0$, and $r = -1/2$ we find

$$(1+1/s)^{-1/2} = 1 - \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{3}{4}\frac{1}{s^2} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{s^3} + \cdots .$$

Going back to (42), we find

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!2}\frac{\Gamma(\frac{2n+1}{2})}{s^{n+3/2}} &= \frac{\sqrt{\pi}}{2}\frac{1}{s^{3/2}}(1+1/s)^{-1/2} \\ &= \frac{\sqrt{\pi}}{2}\frac{1}{ss^2}\frac{1}{(1+1/s)^{1/2}} \\ &= \frac{\sqrt{\pi}}{2}\frac{1}{s\sqrt{1+s}}.\end{aligned}$$

Thus, the Laplace transform is

$$\mathcal{L}\left(\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-p^2}\,dp\right)(s) = \frac{1}{s\sqrt{1+s}}. \tag{43}$$

An alternative method using directly the definition of the Laplace transform and changing the order of integration in multiple integrals, is as follows. From the definition we have:

$$\mathcal{L}\left(\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx\right)(s) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-st}\int_0^{\sqrt{t}} e^{-x^2}\,dx\,dt. \tag{44}$$

The idea is now to change order of integration. Note that x runs from 0 to $\sqrt{t}$ while t runs from 0 to ∞. The curve $x = \sqrt{t}$ is the right part of the curve $t = x^2$ (i.e. the part where $x \geq 0$). Hence in the xt-plane we are integrating over the infinite region above the curve $t = x^2$ with $x \geq 0$. Thus we see that this is the region where t runs from x^2 to ∞ and x runs from 0 to ∞. Hence:

$$\begin{aligned}\mathcal{L}\left(\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx\right) &= \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-x^2}\int_{x^2}^{\infty} e^{-st}\,dt\,dx \\ &= \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-x^2}\frac{e^{-sx^2}}{s}\,dx \\ &= \frac{2}{s\sqrt{\pi}}\int_0^{\infty} e^{-x^2(1+s)}\,dx. \end{aligned}\tag{45}$$

We now use a change of variables $u := x\sqrt{(1+s)}$. Then u runs from 0 to ∞ and $du = \sqrt{1+s}dx$. Substituting this, we therefore finally have

$$\mathcal{L}\left(\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx\right)(s) = \frac{2}{s\sqrt{\pi(s+1)}}\int_0^{\infty} e^{-u^2}\,du = \frac{1}{s\sqrt{s+1}} \tag{46}$$

where the last equality follows from the Gauss integral $I = \int_0^{\infty} e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$ from **d)**.

Finally, we can also proceed as follows. Let $f(t) := \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2}\, dx$. Then by the fundamental theorem of calculus, we have $f'(t) = \frac{2}{\sqrt{\pi}} e^{-t} \frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{\pi}} e^{-t} \frac{1}{\sqrt{t}}$. Note that $f(0) = 0$. Hence by the formula for the Laplace transform of the derivative, we get:

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s). \tag{47}$$

That is $\mathcal{L}(f)(s) = \frac{\mathcal{L}(f')(s)}{s}$. Now, we have:

$$\mathcal{L}(f')(s) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t}}{\sqrt{t}} e^{-st}\, dt = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(s+1)t}}{\sqrt{t}}\, dt. \tag{48}$$

We use a change of variables. Put $u := \sqrt{(1+s)t}$. Then $du = \frac{\sqrt{1+s}}{2\sqrt{t}}\, dt$ and u runs from 0 to ∞. Substituting this, we therefore have:

$$\mathcal{L}(f')(s) = \frac{2}{\sqrt{\pi}\sqrt{1+s}} \int_0^\infty e^{-u^2}\, du = \frac{2}{\sqrt{\pi}\sqrt{1+s}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{1+s}} \tag{49}$$

where the second equality follows from the Gauss integral $I = \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$ from **d)**. Hence we get:

$$\mathcal{L}(f)(s) = \frac{\mathcal{L}(f')(s)}{s} = \frac{1}{s\sqrt{1+s}}. \tag{50}$$