TMA4125 Matematikk 4N

Øving 3 - Fourierrekker I - LF

Obligatoriske oppgaver

1 See the Lecture notes

2 **Matlab:**

```
clear

%her plotter vi heavisidefunksjonen
n=20;
x=-2*pi:.01:2*pi;
y=heaviside(x);
plot(x,y)
axis([-2*pi 2*pi -0.2 1.2])
%vi skrur p  hold on, slik at ikke heaviside blir slettet n r vi plotter de
%neste funksjonene
hold on
%en liten pause for at plottet skal synke inn i hjernene v re
pause

%dette er den f rste fouriermoden, nemlig likevekstlinjen
z=ones(1,length(x))/2;
plot(x,z);
pause(1)
hold off

%her kommer resten av fouriermodene
for i=1:n
    %vi plotter heaviside p  nytt
    plot(x,y)
    hold on
    %trunkert fourierrekke
    z=z+2/(pi*(2*i-1))*sin((2*i-1)*x);
    plot(x,z)
    pause(0.5)
    hold off
end
```

Python:

```
import numpy as np
import matplotlib.pyplot as plt

#fourierrekken skal ha K ledd
K=10

#vi plotter med finhet paa N punkt
N=1000

#t-aksen
```

```
x=np.linspace(-2*np.pi,2*np.pi,num=N)

#heavisidefunksjonen
f=np.zeros(N)
for i in range(N):
    if x[i] > 0:
        f[i]=1

#trunkert fourierrekke
z=np.ones(N)/2.0
for i in range(K):
    z=z+2.0/(np.pi*(2*i+1))*np.sin((2*i+1)*x)

#plotte heaviside og trunkert fourierrekke
plt.plot(x,f)
plt.plot(x,z)

#plotte x-aksen
plt.plot(x,np.zeros(N))

#korrekt utsnitt
plt.axis([-2*np.pi,2*np.pi,-1,2])

#vise plot
plt.savefig('hei')
```

$\boxed{3}$ The function f is neither odd nor even, so we need to compute all the Fourier coefficients a_0, a_n and b_n. We get

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi} x(\pi - x)\,\mathrm{d}x = \frac{1}{2\pi}\left[\frac{\pi}{2}x^2 - \frac{1}{3}x^3\right]_0^{\pi} = \frac{\pi^2}{12}.$$

We need the following integrals to compute a_n and b_n:

$$\int_0^{\pi} x\cos nx\,\mathrm{d}x = \underbrace{\left[\frac{1}{n}x\sin nx\right]_0^{\pi}}_{=0} - \frac{1}{n}\int_0^{\pi}\sin nx\,\mathrm{d}x = \left[\frac{1}{n^2}\cos nx\right]_0^{\pi} = \frac{(-1)^n - 1}{n^2}$$

$$\int_0^{\pi} x\sin nx\,\mathrm{d}x = \left[-\frac{1}{n}x\cos nx\right]_0^{\pi} + \frac{1}{n}\underbrace{\int_0^{\pi}\cos nx\,\mathrm{d}x}_{=0} = -\frac{\pi}{n}(-1)^n$$

$$\int_0^{\pi} x^2\cos nx\,\mathrm{d}x = \underbrace{\left[\frac{1}{n}x^2\sin nx\right]_0^{\pi}}_{=0} - \frac{2}{n}\underbrace{\int_0^{\pi} x\sin nx\,\mathrm{d}x}_{=-\frac{\pi}{n}(-1)^n} = \frac{2\pi}{n^2}(-1)^n$$

$$\int_0^{\pi} x^2\sin nx\,\mathrm{d}x = \left[-\frac{1}{n}x^2\cos nx\right]_0^{\pi} + \frac{2}{n}\underbrace{\int_0^{\pi} x\cos nx\,\mathrm{d}x}_{=\frac{(-1)^n-1}{n^2}}$$

$$= -\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}\left[(-1)^n - 1\right].$$

We thus get

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x(\pi - x)\cos nx \,\mathrm{d}x \\
&= \int_0^{\pi} x\cos nx \,\mathrm{d}x - \frac{1}{\pi}\int_0^{\pi} x^2\cos nx \,\mathrm{d}x = \frac{(-1)^n - 1}{n^2} - \frac{2}{n^2}(-1)^n \\
&= \begin{cases} -\frac{2}{n^2} & n \text{ even} \\ 0 & n \text{ odd,} \end{cases} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x(\pi - x)\sin nx \,\mathrm{d}x \\
&= \int_0^{\pi} x\sin nx \,\mathrm{d}x - \frac{1}{\pi}\int_0^{\pi} x^2\sin nx \,\mathrm{d}x \\
&= -\frac{\pi}{n}(-1)^n - \frac{1}{\pi}\left[-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}\left[(-1)^n - 1\right]\right] = \begin{cases} \frac{4}{n^3\pi} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}
\end{aligned}
$$

The Fourier series of $f(x)$ is thus

$$
f(x) = \frac{\pi^2}{12} - \sum_{n=1}^{\infty} \frac{1}{2n^2}\cos 2nx + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\sin[(2n-1)x].
$$

For the second part, evaluating the Fourier series above wisely at $x = 0$ we get

$$
f(0) = \frac{\pi^2}{12} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}.
$$

Since we also have $f(0) = 0$, we find

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
$$

Matlab:

```
clear

%x-koordinater
n=50;
x = linspace(-pi,pi,n);

%funksjonen
f = @(x) x.*(pi-x) .* heaviside(x);

y=f(x);
plot(x,y)
axis([-pi pi , -0.5, 3]);

%vi skrur p  hold on, slik at ikke heaviside blir slettet n r vi plotter de
%neste funksjonene
hold on
%en liten pause for at plottet skal synke inn i hjernene v re
pause(2);

%dette er den f rste fouriermoden, nemlig likevekstlinjen
```

```
z=ones(1,length(x))*pi^2/12;
plot(x,z);
pause(1);
hold off

%her kommer resten av fouriermodene
m = 3; %antall moder
for i=1:m
    %vi plotter funksjonen p   nytt
    plot(x,y)
    axis([-pi pi , -0.5, 3]);
    hold on
    %trunkert fourierrekke
    z = z - cos(2*i*x)/(2*i^2) + (4/pi)*sin((2*i-1)*x)/((2*i-1)^3)
    plot(x,z)
    pause(2)
    hold off
end
```

Anbefalte oppgaver

$\boxed{1}$ First note that $f(x)$ is an even function, which implies that $b_n = 0$ for all n. The coefficient a_0 is given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,\mathrm{d}x = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}.$$

Finally, using the trigonometric identity

$$\sin[(n+1)x] - \sin[(n-1)x] = 2\sin x\cos nx,$$

we get

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi}\sin[(n+1)x] - \sin[(n-1)x]\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[-\frac{\cos[(n+1)x]}{n+1} + \frac{\cos[(n-1)x]}{n-1}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) = -\frac{2}{\pi}\frac{(-1)^n+1}{n^2-1} \\
&= \begin{cases} -\frac{4}{\pi}\frac{1}{n^2-1} & n \text{ even} \\ 0 & n \text{ odd,} \end{cases}
\end{aligned}$$

where we have used $\cos n\pi = (-1)^n$. The Fourier series of $f(x)$ is thus given by

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx.$$

For the second part, evaluating the Fourier series above wisely at $x = 0$ we get

$$f(0) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}.$$

Since we also have $f(0) = 0$, we find

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

$\boxed{2}$ First note that the function f has period 2π. The Fourier coefficients a_n and b_n of a function with period 2π is given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,\mathrm{d}x \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,\mathrm{d}x. \end{aligned}$$

Using the orthogonality property of trigonometric system, some work can be spared by extracting these coefficients directly from the function:

$$\begin{aligned} a_0 &= 5 \\ a_n &= \begin{cases} -4 & n = 2 \\ 5 & n = 8 \\ 0 & \text{otherwise} \end{cases} \\ b_n &= \begin{cases} -2 & n = 5 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$\boxed{3}$ The function $g(x) = f(3x)$ has period 2π and can therefore be written as

$$g(x) = \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos nx + \tilde{b}_n \sin nx,$$

where the coefficients $\tilde{a}_0$, $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ are to be determined. From the equality $g(x) = f(3x)$ and using the Fourier series of f we get

$$\tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos nx + \tilde{b}_n \sin nx = g(x) = f(3x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 3nx + b_n \sin 3nx.$$

The expressions on the left hand side and the right hand are equal whenever

$$\begin{aligned} \tilde{a}_0 &= a_0 \\ \tilde{a}_n &= \begin{cases} a_{n/3} & \text{whenever } n \text{ is divisible by } 3 \\ 0 & \text{otherwise} \end{cases} \\ \tilde{b}_n &= \begin{cases} b_{n/3} & \text{whenever } n \text{ is divisible by } 3 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and this determines the Fourier series of g.