Øving 4 - Fourierrekker - LF

Obligatoriske oppgaver

1 See the Lecture notes

2 **Matlab:**

```
clear
%se oving_1.m hvis du lurer p   noe her. merk hvordan surfplot er enklere i
%matlab enn python.
x=0:.1:pi;
y=0:.1:pi;
%merk den snasne bruken av ytreprodukt
u=sinh(y)'*sin(x);

%plot
surf(x,y,u)
axis([0 pi 0 pi 0 15])
```

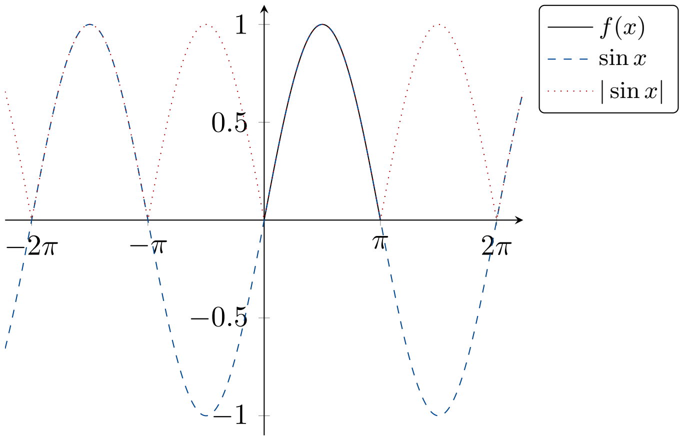
Python:

```
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter
import numpy as np

#antall punkt
N=100

#x- og y-aksen
x=np.linspace(0,np.pi,num=N)
y=np.linspace(0,np.pi,num=N)

#ytreprodukt av x- og y-aksen. surf trenger dette for    funke.
X,Y=np.meshgrid(x,y)

#funksjonsverdiene
u=np.multiply(np.sinh(X),np.sin(Y))

#lage plot. et par av disse kommandoene vet jeg ikke hva g j r , men jeg ...
    fant dem p   nettet
fig = plt.figure()
ax = fig.gca(projection='3d')
surf = ax.plot_surface(X, Y, u, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False)

#korrekt utsnitt av xy-planet
plt.axis([0,np.pi,0,np.pi])

# navn p   aksene
```

```
plt.xlabel(r'$x$')
plt.ylabel(r'$y$')
ax.set_zlabel(r'$u(x,y)$')

#vise plot
plt.show()
```

3 Periodic extensions

The odd periodic extension of $f(x)$ is simply $\sin(x)$, since $\sin(x)$ is already an odd function. The even periodic extensions of $f(x)$ is $|\sin x|$. A sketch of these functions can be found in Figure 1.



Figur 1: Sketch of the odd ($\sin x$) and even ($|\sin x|$) periodic extensions of $f(x)$.

Fourier series

Odd periodic extension: We want to find the Fourier series of $\sin(x)$. The function is odd, so it follows that $a_0 = a_1 = \cdots = 0$. We thus look for coefficients $\{b_n\}$ such that

$$\sin x = \sum_{n=1}^{\infty} b_n \sin nx = b_1 \cdot \sin x + b_2 \cdot \sin 2x + b_3 \cdot \sin 3x + \ldots$$

The two sides are equal whenever $b_1 = 1$ and $b_2 = b_3 = b_4 = \cdots = 0$ (The Fourier series of $\sin(x)$ is simply $\sin(x)$).

Even periodic extension: We want to find the Fourier series of $|\sin(x)|$. We have $b_1 = b_2 = \cdots = 0$ since the function is even. We compute

$$a_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi \sin x\,\mathrm{d}x = \frac{1}{\pi}\left[-\cos x\right]_0^\pi = \frac{2}{\pi}.$$

Further, using the formula

$$\sin x \cos nx = \frac{1}{2}\left\{\sin[(1+n)x] + \sin[(1-n)x]\right\},$$

we compute for $n > 1$

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2}{\pi}\int_0^\pi \sin x\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\pi \sin[(1+n)x] + \sin[(1-n)x]\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[-\frac{\cos[(1+n)x]}{1+n} - \frac{\cos[(1-n)x]}{1-n}\right]_0^\pi \\
&= \frac{(-1)^n+1}{\pi}\left(\frac{1}{1+n}+\frac{1}{1-n}\right) = \begin{cases} \frac{4}{\pi}\frac{1}{1-n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.
\end{aligned}
$$

The special case $n = 1$ must be considered separately

$$
a_1 = \frac{1}{\pi}\int_0^\pi \sin x\cos x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^\pi \sin 2x\,\mathrm{d}x = \frac{1}{2\pi}\left[-\frac{1}{2}\cos 2x\right]_0^\pi = 0
$$

Thus, the Fourier cosine series of $|\sin(x)|$ is given by

$$
f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^\infty \frac{1}{1-4n^2}\cos 2nx.
$$

$\boxed{4}$ As f is an even function, $b_n = 0$ for all n. With $L = 1$ we find

$$
a_0 = \frac{1}{2L}\int_{-L}^L f(x)\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3},
$$

and

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2}{1}\int_0^1 x^2\cos n\pi x\,\mathrm{d}x \\
&= 2\left(\left[\frac{1}{n\pi}x^2\sin n\pi x\right]_0^1 - \frac{2}{n\pi}\int_0^1 x\sin n\pi x\,\mathrm{d}x\right) \\
&= -\frac{4}{n\pi}\left(\left[-\frac{1}{n\pi}x\cos n\pi x\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\,\mathrm{d}x\right) = \frac{4(-1)^n}{n^2\pi^2}.
\end{aligned}
$$

Recall Parseval's identity

$$
2a_0^2 + \sum_{n=1}^\infty \left(a_n^2 + b_n^2\right) = \frac{1}{L}\int_{-L}^L f(x)^2\,\mathrm{d}x.
$$

Thus,

$$
\frac{2}{9} + \sum_{n=1}^\infty \frac{16}{n^4\pi^4} = \int_{-1}^1 x^4\,\mathrm{d}x = 2\int_0^1 x^4\,\mathrm{d}x = 2\left[\frac{1}{5}x^5\right]_0^1 = \frac{2}{5},
$$

which implies the result

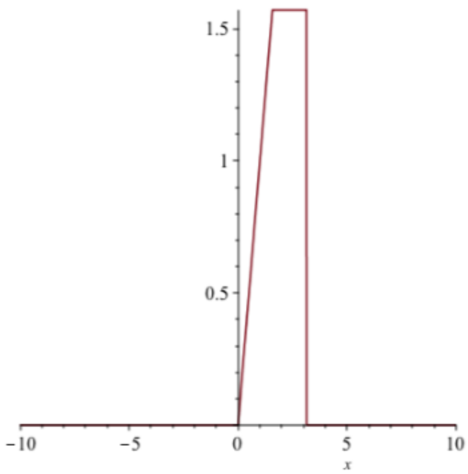
$$
\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.
$$

Anbefalte oppgaver

$\boxed{1}$ The function f is given by

$$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi/2, & \pi/2 < x < \pi, \end{cases}$$

and its graph looks like this:



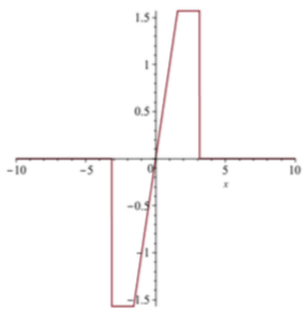
Figur 2: The function f.

Odd extension.

The odd extension f_o satisfies $f_o(-x) = -f(x)$. Hence we get

$$f_o(x) = \begin{cases} -\pi/2, & -\pi < x < -\pi/2 \\ x, & -\pi/2 < x < 0 \\ x, & 0 < x < \pi/2 \\ \pi/2, & \pi/2 < x < \pi \end{cases}$$

Its graph looks like this



Figur 3: f_o on the interval $(-\pi, \pi)$.

Fourier series.

Recall the formula for the Fourier sine series of f_o:

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

where:

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_0(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx).$$

The last equality follows by definition of $f_o(x) = f(x)$ for $\pi > x > 0$. We use the definition of f and split the integral. We get:

$$\frac{\pi}{2}b_n = \int_0^{\pi} f(x)\sin(nx)\,dx = \int_0^{\pi/2} x\sin(nx)\,dx + \int_{\pi/2}^{\pi}\frac{\pi}{2}\sin(nx)\,dx.$$

We will find it convenient to use the following formula (can be deduced for instance by integration by parts):

$$\int x\sin(kx)\,dx = \frac{\sin(kx) - kx\cos(kx)}{k^2} + C.$$

Using this we get:

$$\begin{aligned}
\frac{\pi}{2}b_n &= \frac{\sin\left(\frac{\pi}{2}n\right) - n\frac{\pi}{2}\cos\left(\frac{\pi}{2}n\right)}{n^2} + \frac{\pi}{2n}\left(-\cos(n\pi) + \cos\left(\frac{\pi}{2}n\right)\right)\\
&= \frac{\sin\left(\frac{\pi}{2}n\right)}{n^2} - \frac{\pi}{2n}\cos\left(\frac{\pi}{2}n\right) - \frac{\pi}{2n}\cos(n\pi) + \frac{\pi}{2n}\cos\left(\frac{\pi}{2}n\right)\\
&= \frac{\sin\left(\frac{\pi}{2}n\right)}{n^2} - \frac{\pi}{2n}(-1)^n.
\end{aligned}$$

In the last equality, we have used that $\cos(n\pi) = (-1)^n$. We now split in even and odd n's. For even n's, we can write $n = 2m$ with $m = 1, 2, \ldots$ which gives:

$$\frac{\pi}{2}b_{2m} = \frac{\sin(\pi m)}{4m^2} - \frac{\pi}{4m}(-1)^{2m} = -\frac{\pi}{4m}$$

and hence:

$$b_{2m} = -\frac{1}{2m}.$$

For n odd, we can write $n = 2m-1$ with $m = 1, 2, \ldots$. This then gives:

$$\begin{aligned}
\frac{\pi}{2}b_{2m-1} &= \frac{\sin\left(\frac{\pi}{2}(2m-1)\right)}{(2m-1)^2} - \frac{\pi}{2(2m-1)}(-1)^{2m-1}\\
&= \frac{\sin\left(m\pi\frac{\pi}{2}\right)}{(2m-1)^2} + \frac{\pi}{2(2m-1)}.
\end{aligned}$$

We use the addition formula for sine:

$$\begin{aligned}\sin(m\pi - \pi/2) &= \sin(m\pi)\cos(\pi/2) - \sin(\pi/2)\cos(m\pi) \\ &= 0 - \cos(m\pi) \\ &= -(-1)^m \\ &= (-1)^{m+1}\end{aligned}$$

Hence:

$$b_{2m-1} = \frac{2(-1)^{m+1}}{\pi(2m-1)^2} + \frac{1}{2m-1}.$$

Thus we get the following Fourier sine series for f_o:

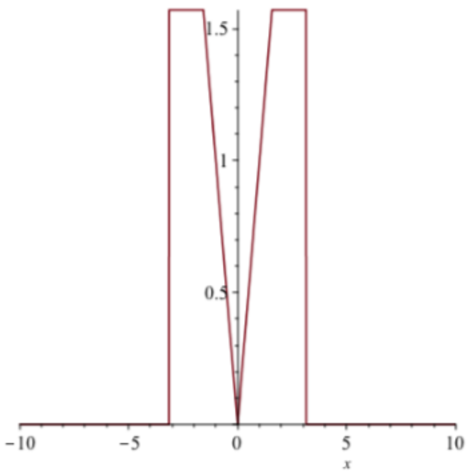
$$f_o(x) = \sum_{m=1}^{\infty} \left\{ \left[\frac{2(-1)^{m+1}}{\pi(2m-1)^2} + \frac{1}{2m-1} \right] \sin\left((2m-1)x\right) - \frac{1}{2m}\sin(2mx) \right\}.$$

Even extension.

The even extension f_e satisfies $f_e(-x) = f(x)$. Hence we get

$$f_e(x) = \begin{cases} \pi/2, & -\pi < x < -\pi/2 \\ -x, & -\pi/2 < x < 0 \\ x, & 0 < x < \pi/2 \\ \pi/2, & \pi/2 < x < \pi, \end{cases}$$

and f_e can be drawn as shown below.



Figur 4: The graph of f_e in the interval $(-\pi, \pi)$

Fourier series.

We recall the formula for the Fourier cosine series of f_e:

$$f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

where:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f_e(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$
$$a_n = \frac{2}{\pi}\int_0^{\pi} f_e(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx.$$

We use the definition of f and split the integrals. This gives for a_0:

$$\begin{aligned}\pi a_0 &= \int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} \frac{\pi}{2}\,dx\\ &= \frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{\pi^2}{4}\\ &= \frac{\pi^2}{8} + \frac{\pi^2}{4}\\ &= \frac{3\pi^2}{8},\end{aligned}$$

thus:

$$a_0 = \frac{3\pi}{8}.$$

For a_n, we get:

$$\frac{\pi}{2}a_n = \int_0^{\pi/2} x\cos(nx) + \frac{\pi}{2}\int_{\pi/2}^{\pi} \cos(nx)\,dx.$$

For the first integral, we use the following formula (can be deduced for instance by integration by parts):

$$\int x\cos(kx)\,dx = \frac{kx\sin(kx)+\cos(kx)}{k^2} + C.$$

Using this, we therefore get:

$$\begin{aligned}\frac{\pi}{2}a_n &= \frac{\frac{\pi}{2}n\sin\left(\frac{\pi}{2}n\right)+\cos\left(\frac{\pi}{2}n\right)}{n^2} - \frac{1}{n^2} + \frac{\pi}{2n}\left(\sin(n\pi) - \sin\left(\frac{\pi}{2}n\right)\right)\\ &= \frac{\pi}{2n}\sin\left(\frac{\pi}{2}n\right) + \frac{\cos\left(\frac{\pi}{2}n\right)}{n^2} - \frac{1}{n^2} - \frac{\pi}{2n}\sin\left(\frac{\pi}{2}n\right)\\ &= \frac{\cos\left(\frac{\pi}{2}n\right)}{n^2} - \frac{1}{n^2}.\end{aligned}$$

For n odd, that is, for $n = 2m-1, m = 1,2,\ldots$, we find $\cos(\pi n/2) = 0$ and hence:

$$a_{2m-1} = -\frac{2}{\pi(2m-1)^2}.$$

For n even, that is, for $n = 2m, m = 1,2,\ldots,$. we have $\cos(\pi n/2) = (-1)^m$ and so we get:

$$a_{2m} = \frac{2}{4\pi m^2}\left((-1)^m - 1\right).$$

For m even, this is 0, while for m odd, this is $-\frac{1}{\pi m^2}$. Thus we only get a non-zero contribution in the case that $m = 1, 3, 5, 7, \ldots$. Because n is even, this gives $n = 2, 6, 10, 14, \ldots$. Thus we can write $n = 2(2m-1)$, with $m = 1, 2, \ldots$. Hence we get:

$$a_{n_{\text{even}}} = a_{2(2m-1)} = -\frac{1}{\pi(2m-1)^2}$$

Thus we finally have the Fourier cosine series of f_e:

$$f_e(x) = \frac{3\pi}{8} + \sum_{m=1}^{\infty} \left\{ -\frac{2}{\pi(2m-1)^2} \cos((2m-1)x) - \frac{1}{\pi(2m-1)^2} \cos(2(2m-1)x) \right\}.$$

$\boxed{2}$ As f is an even function, $b_n = 0$ for all n. With $L = \frac{1}{2}$ we find

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,\mathrm{d}x = 2 \int_0^{\frac{1}{2}} \cos \pi x \,\mathrm{d}x = \left[\frac{1}{\pi} \sin \pi x \right]_0^{\frac{1}{2}} = \frac{2}{\pi}$$

and, using the following formula

$$\cos \pi x \cos 2n\pi x = \frac{1}{2} \left\{ \cos[\pi(1+2n)x] + \cos[\pi(1-2n)x] \right\},$$

we have

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,\mathrm{d}x = 4 \int_0^{\frac{1}{2}} \cos \pi x \cos 2n\pi x \,\mathrm{d}x \\ &= 2 \int_0^{\frac{1}{2}} \cos[\pi(1+2n)x] + \cos[\pi(1-2n)x]\,\mathrm{d}x \\ &= \frac{2}{\pi} \left[\frac{\sin[\pi(1+2n)x]}{1+2n} + \frac{\sin[\pi(1-2n)x]}{1-2n} \right]_0^{\frac{1}{2}} \\ &= \frac{2\cos \pi n}{\pi} \left(\frac{1}{1+2n} + \frac{1}{1-2n} \right) = \frac{4}{\pi} \frac{(-1)^n}{1-4n^2}. \end{aligned}$$

Thus, the Fourier series of $f(x)$ is

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2} \cos 2n\pi x.$$

$\boxed{3}$ We found the Fourier coefficients a_0 and a_n in a previous exercise. Thus, for $-1 \le x \le 1$,

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} (-1)^n \frac{\cos(n\pi x)}{n^2}.$$

Integrating both sides from $-1/2$ to $1/2$ (not from -1 to 1!), we get

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{3} dx + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(n\pi x)\,dx \tag{1}$$

The right integral is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(n\pi x)\,dx = \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1/2}^{1/2} = \frac{1}{n\pi}\Big[\sin\big(\frac{n\pi}{2}\big) - \sin\big(\frac{-n\pi}{2}\big)\Big].$$

If we denote $f(n) := \frac{1}{2}\Big[\sin\big(\frac{n\pi}{2}\big) - \sin\big(\frac{-n\pi}{2}\big)\Big]$ it is clear that

$$f(n) = \begin{cases} 0, \text{ for } n = 2,4,6,8,\dots \\ 1 \text{ for } n = 1,5,9,\dots \\ -1 \text{ for } n = 3,7,11,\dots \end{cases}$$

Note that $f(n)$ is zero for even numbers. Thus, summing over odd numbers (changing n to $2n-1$) we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} f(n) = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{(2n-1)^3} f(2n-1) = \sum_{n=1}^{\infty} \frac{(-1)}{(2n-1)^3}(-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

Returning to (1) we have

$$\frac{2}{3}\Big(\frac{1}{2}\Big)^3 = \frac{1}{3} + \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

Thus, by algebraic manipulations

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{(2n-1)^3} = \frac{\pi^3}{8}\left(\frac{1}{3} - \frac{1}{12}\right) = \frac{\pi^3}{32}.$$