TMA4125 Matematikk 4N

Øving 5 - Fouriertransform - LF

Obligatoriske oppgaver

1 See the notes ¯_(ツ)_/¯

2 **Matlab:**

```
%x- og t-aksen
x=0:.1:pi;
t=0:pi/100:2*pi;

%sette opp funksjon og plotte hver frame
for j=1:length(t)
    %funksjonsverdier p   innev rende tidspunkt
    f=sin(1*t(j))*sin(1*x);
    %vi plotter disse funksjonsverdiene
    plot(x,1*f)
    axis([0 pi -1.2 1.2])
    pause(.03)
end
```

Python:

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation


#ok jeg kryper til korset, og innroemmer at jeg ikke skjonner baret av ...
    hvordan denne koden funker. jeg fant den paa nettet, og greide aa ...
    modifisere den akkurat nok til at jeg fikk ut den animasjonen jeg ville ha
fig, ax = plt.subplots()
ax.set_xlabel(r'$x$')
ax.set_xlabel(r'$y$')
xdata, ydata = [], []
ln, = plt.plot([], [], animated=True,label=r'$u\, (x,t)$') # skriv inn 'ro' ...
    som tredje argument hvis du vil ha tilbake punkter i stedet for linjer.

def init():
    ax.set_xlim(0, 2*np.pi)
    ax.set_ylim(-1.1, 1.1)
    return ln,

def update(frame):
    xdata=np.linspace(0, 2*np.pi, 128)
    ydata=np.sin(frame)*np.sin(np.linspace(0, np.pi, 128))
    ln.set_data(xdata, ydata)
    return ln,

# Man kan endre hastigheten ved aa endre paa interval-parameteren
```

```
ani = FuncAnimation(fig, update, frames=np.linspace(0, 2*np.pi, 128), ...
    interval=25,
                     init_func=init, blit=True)
plt.legend()
plt.show()
```

3 By the definition of Fourier transform we have

$$\begin{aligned}\hat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}wx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-|x|}\mathrm{e}^{-\mathrm{i}wx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-|x|}\cos wx\,\mathrm{d}x - \frac{\mathrm{i}}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} \mathrm{e}^{-|x|}\sin wx\,\mathrm{d}x}_{=0} \\ &= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \mathrm{e}^{-|x|}\cos wx\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \mathrm{e}^{-x}\cos wx\,\mathrm{d}x \\ &= \sqrt{\frac{2}{\pi}}\left[\frac{\mathrm{e}^{-x}}{1+w^2}(-\cos wx + w\sin wx)\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}.\end{aligned}$$

By the definition of inverse Fourier transform we get

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\mathrm{e}^{\mathrm{i}wx}\,\mathrm{d}w = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+w^2}\mathrm{e}^{\mathrm{i}wx}\,\mathrm{d}w.$$

By evaluating this equation wisely at $x = 0$ we get

$$1 = f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+w^2}\,\mathrm{d}w = \frac{2}{\pi}\int_{0}^{\infty} \frac{1}{1+w^2}\,\mathrm{d}w.$$

Thus,

$$\int_0^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

4 Let $g(x) = \mathrm{e}^{-x^2}$. By the differentiation rule, we have

$$\mathscr{F}\{g''(x)\} = -w^2\,\mathscr{F}\{g(x)\}.$$

Since

$$g''(x) = -2\mathrm{e}^{-x^2} + 4x^2\mathrm{e}^{-x^2} = -2g(x) + 4f(x),$$

and using the linearity of the Fourier transform, we get

$$4\,\mathscr{F}\{f(x)\} = 2\,\mathscr{F}\{g(x)\} + \mathscr{F}\{g''(x)\} = 2\,\mathscr{F}\{g(x)\} - w^2\,\mathscr{F}\{g(x)\}.$$

The Fourier transform of $g(x)$ is known to be

$$\mathscr{F}\{g(x)\} = \frac{1}{\sqrt{2}}\mathrm{e}^{-w^2/4},$$

so that

$$\hat{f}(w) = \mathscr{F}\{f(x)\} = \frac{2-w^2}{4\sqrt{2}}\mathrm{e}^{-w^2/4}.$$

5 We take the Fourier transform of the convolution:

$$\mathscr{F}\{f * g\} = \sqrt{2\pi}\,\mathscr{F}\{f\}\,\mathscr{F}\{g\}.$$

Notice that $g(x) = -\frac{1}{2}f'(x)$, such that (using the derivative of the Fourier transform)

$$\mathscr{F}\{g\} = -\frac{1}{2}\,\mathscr{F}\{f'\} = -\frac{\mathrm{i}w}{2}\,\mathscr{F}\{f\}.$$

Using this and $\mathscr{F}\{f\} = \frac{1}{\sqrt{2}}\mathrm{e}^{-w^2/4}$, we get

$$\mathscr{F}\{f * g\} = \sqrt{2\pi}\frac{1}{\sqrt{2}}\mathrm{e}^{-w^2/4} \cdot \left(-\frac{\mathrm{i}w}{2}\frac{1}{\sqrt{2}}\mathrm{e}^{-w^2/4}\right) = -\frac{\mathrm{i}w\sqrt{\pi}}{2\sqrt{2}}\mathrm{e}^{-w^2/2}$$

Taking the inverse transform yields

$$(f * g)(x) = -\frac{i}{4}\int_{-\infty}^{\infty} w\mathrm{e}^{-w^2/2}\mathrm{e}^{\mathrm{i}wx}\,\mathrm{d}w.$$

Anbefalte oppgaver

1 **Fourier transform.**

By the definition of Fourier transform we have

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}wx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \sin x\mathrm{e}^{-\mathrm{i}wx}\,\mathrm{d}x \\
&= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \sin x\cos wx\,\mathrm{d}x}_{=0} - \frac{\mathrm{i}}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \sin x\sin wx\,\mathrm{d}x \\
&= -\mathrm{i}\sqrt{\frac{2}{\pi}}\int_{0}^{\pi} \sin x\sin wx\,\mathrm{d}x
\end{aligned}$$

Consider first $w = \pm 1$

$$\begin{aligned}
\int_0^{\pi} \sin x\sin wx\,\mathrm{d}x &= \pm\int_0^{\pi} \sin^2 x\,\mathrm{d}x = \pm\int_0^{\pi} \frac{1-\cos 2x}{2}\,\mathrm{d}x \\
&= \pm\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi} = \pm\frac{\pi}{2}.
\end{aligned}$$

Now, we use the following trigonometric identities

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

and

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

If $w \neq \pm 1$, we get

$$\begin{aligned}\int_0^\pi \sin x \sin wx \,\mathrm{d}x &= \int_0^\pi \frac{1}{2}(\cos[(1-w)x] - \cos[(1+w)x])\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{\sin[(1-w)x]}{1-w} - \frac{\sin[(1+w)x]}{1+w}\right]_0^\pi \\ &= \frac{1}{2}\left(\frac{\sin[(1-w)\pi]}{1-w} - \frac{\sin[(1+w)\pi]}{1+w}\right) \\ &= \frac{1}{2}\left(\frac{\sin(\pi w)}{1-w} + \frac{\sin(\pi w)}{1+w}\right) = \frac{\sin \pi w}{1-w^2}.\end{aligned}$$

Thus,

$$\hat{f}(w) = \begin{cases} -\mathrm{i}\sqrt{\frac{2}{\pi}}\frac{\sin \pi w}{1-w^2} & w \neq \pm 1 \\ \mathrm{i}\sqrt{\frac{\pi}{2}} & w = -1 \\ -\mathrm{i}\sqrt{\frac{\pi}{2}} & w = 1. \end{cases}$$

The integral.

By the definition of inverse Fourier transform we get

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(w)\mathrm{e}^{\mathrm{i}wx}\,\mathrm{d}w = -\frac{\mathrm{i}}{\pi}\int_{-\infty}^\infty \frac{\sin \pi w}{1-w^2}\mathrm{e}^{\mathrm{i}wx}\,\mathrm{d}w \\ &= -\frac{\mathrm{i}}{\pi}\underbrace{\int_{-\infty}^\infty \frac{\sin \pi w}{1-w^2}\cos wx\,\mathrm{d}w}_{=0} - \frac{\mathrm{i}^2}{\pi}\int_{-\infty}^\infty \frac{\sin \pi w}{1-w^2}\sin wx\,\mathrm{d}w \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sin \pi w}{1-w^2}\sin wx\,\mathrm{d}w. \end{aligned}$$

By evaluating this equation wisely at $x = \frac{\pi}{2}$ we get

$$1 = f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\int_{-\infty}^\infty \frac{\sin(\pi w)\sin(\pi w/2)}{1-w^2}\,\mathrm{d}w.$$

Thus,

$$\int_0^\infty \frac{\sin(\pi w)\sin(\pi w/2)}{1-w^2}\,\mathrm{d}w = \frac{\pi}{2}.$$

2 Recall that the Fourier transform of f is given by

$$\mathcal{F}(f)(w) := \hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-ixw}\,dx.$$

a) Now

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{else} \end{cases}$$

gives

$$\sqrt{2\pi}\hat{f}(w) = \int_0^\infty e^{-x(iw+1)}\,dx = \frac{1}{1+iw}.$$

Thus

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}(1+iw)}.$$

b) Now

$$f(x) = \begin{cases} 1-x^2, & x \in [-1,1] \\ 0, & \text{else} \end{cases}$$

gives

$$\begin{aligned}
\sqrt{2\pi}\hat{f}(w) &= \int_{-1}^{1}(1-x^2)e^{-iwx}\,dx = \int_{-1}^{1}(1-x^2)\left(\frac{e^{-iwx}}{-iw}\right)'\,dx \\
&= 0 - \int_{-1}^{1}(1-x^2)'\left(\frac{e^{-iwx}}{-iw}\right)\,dx = \int_{-1}^{1}2x\left(\frac{e^{-iwx}}{-iw}\right)\,dx \\
&= \int_{-1}^{1}2x\left(\frac{e^{-iwx}}{(-iw)^2}\right)'\,dx = 2x\frac{e^{-iwx}}{(-iw)^2}\Big|_{-1}^{1} - \int_{-1}^{1}2\left(\frac{e^{-iwx}}{(-iw)^2}\right)\,dx \\
&= \frac{2e^{-iw}+2e^{iw}}{-w^2} - \int_{-1}^{1}2\left(\frac{e^{-iwx}}{(-iw)^3}\right)'\,dx \\
&= \frac{4\cos w}{-w^2} - \frac{2e^{-iwx}}{(-iw)^3}\Big|_{-1}^{1} \\
&= \frac{4\cos w}{-w^2} - \frac{2(e^{-iw}-e^{iw})}{iw^3} = \frac{4\cos w}{-w^2} + \frac{4\sin w}{w^3}.
\end{aligned}$$

Thus

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\left(\frac{4\cos w}{-w^2} + \frac{4\sin w}{w^3}\right).$$

c) Now

$$f(x) = \begin{cases} T+x, & -T \le x < 0 \\ T-x, & 0 \le x < T \\ 0, & \text{else} \end{cases}$$

gives

$$\begin{aligned}
\sqrt{2\pi}\hat{f}(w) &= \int_{-T}^{0}(T+x)e^{-iwx}\,dx + \int_{0}^{T}(T-x)e^{-iwx}\,dx \\
&= \int_{-T}^{0}(T+x)\left(\frac{e^{-iwx}}{-iw}\right)'\,dx + \int_{0}^{T}(T-x)\left(\frac{e^{-iwx}}{-iw}\right)'\,dx \\
&= \frac{T}{-iw} - 0 - \int_{-T}^{0}\left(\frac{e^{-iwx}}{-iw}\right)\,dx + 0 - \frac{T}{-iw} + \int_{0}^{T}\left(\frac{e^{-iwx}}{-iw}\right)\,dx \\
&= \int_{0}^{T}\left(\frac{e^{-iwx}}{-iw}\right)\,dx - \int_{-T}^{0}\left(\frac{e^{-iwx}}{-iw}\right)\,dx \\
&= \frac{e^{-iwx}}{(-iw)^2}\Big|_{0}^{T} - \frac{e^{-iwx}}{(-iw)^2}\Big|_{-T}^{0} \\
&= \frac{2-e^{iTw}-e^{-iTw}}{w^2} = \frac{2-2\cos Tw}{w^2}.
\end{aligned}$$

Thus

$$\hat{f}(w) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos Tw}{w^2}.$$