TMA4125 Matematikk 4N

Øving 6 - Bølgeligningen - LF

Obligatoriske oppgaver

1 See Theorem 4.2 in the notes

2 **Matlab:**

```
%se oving_5.m for kommentarer
x=0:.1:3;
t=0:pi/100:2*pi;


for j=1:length(t)
    f=1/pi*sin(pi*t(j))*sin(pi*x)+1/(4*pi)*sin(2*pi*t(j))*sin(2*pi*x);
    plot(x,f)
    axis([0 3 -1.2 1.2])
    pause(.03)
end
```

Python:

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation

#jeg skjoenner ikke hvordan denne koden funker heller. det er jo den samme ...
    som i oving5.py
fig, ax = plt.subplots()
xdata, ydata = [], []
ax.set_xlabel(r'$x$')
ax.set_xlabel(r'$y$')
ln, = plt.plot([], [], animated=True,label=r'$u\, (x,t)$') # skriv inn 'ro' ...
    som tredje argument hvis du vil ha tilbake punkter i stedet for linjer.

def init():
    ax.set_xlim(0, 3)
    ax.set_ylim(-1, 1)
    return ln,

def update(frame):
    xdata=np.linspace(0, 2*np.pi, 128)
    ydata=1/np.pi*np.sin(np.pi*frame)*np.sin(np.pi*xdata)+1/(4*np.pi)*np.sin(2*np.pi*frame)
    ln.set_data(xdata, ydata)
    return ln,

# Man kan endre hastigheten ved    endre p    interval-parameteren
ani = FuncAnimation(fig, update, frames=np.linspace(0, 2*np.pi, 128), ...
    interval=35,
                    init_func=init, blit=True)
plt.legend()
plt.show()
```

$\boxed{3}$ We first find the solution of the general problem

$$u_{tt} = cu_{xx}, \quad 0 < x < L, \quad t > 0, \tag{1}$$

by seperation of variables. Then we insert our special case of c and L, boundary conditions and initial conditions. Separation of variables turns the equation (1) into

$$F(x)G''(t) = c^2 F''(x)G(t).$$

Rearranging gives

$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)}.$$

Since this holds for all x and t, both sides must be equal to a constant k, so we have the two equations

$$F'' - kF = 0 \tag{2}$$

$$G'' - c^2 kG = 0. \tag{3}$$

The function F.

We start by investigating the function F given by (2).

Boundary conditions.

The types of boundary conditions we are concerned with are

$$u(0,t) = u(L,t) = 0, \qquad \text{(Dirichlet boundary conditions)},$$

and

$$u_x(0,t) = u_x(L,t) = 0, \qquad \text{(Neumann boundary conditions)}.$$

By inserting $u(x,t) = F(x)G(t)$ into the boundary conditions, we get conditions on F:

$$F(x)|_{x=0} = 0 \quad \text{and} \quad F(x)|_{x=L} = 0, \tag{4}$$

and

$$\left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=L} = 0. \tag{5}$$

From earlier courses we know that there are three types of solutions of (2), depending on whether k is positive, negative or zero:

1. If $k = 0$, then $F(x) = A + Bx$ for constants A and B. For the boundary conditions in (4) and (5) to be satisfied, we must have $A = B = 0$ (trivial solution) and $B = 0$, respectively. In the second case this yields a constant solution $F_0 = A$.

2. If $k > 0$, then
$$F(x) = Ce^{\sqrt{k}x} + De^{-\sqrt{k}x}.$$
and

$$F'(x) = C\sqrt{k}e^{\sqrt{k}x} - D\sqrt{k}e^{-\sqrt{k}x},$$

We see that the only way that the boundary conditions $F(0) = F(L) = 0$ or $F'(0) = F'(L) = 0$ are fulfilled is if $C = D = 0$. This is the trivial solution, and is of no interest.

3. If $k < 0$, then

$$F(x) = M\cos(\sqrt{-k}x) + N\sin(\sqrt{-k}x).$$

and

$$F'(x) = -\sqrt{-k}M\sin(\sqrt{-k}x) + \sqrt{-k}N\cos(\sqrt{-k}x).$$

Dirichlet boundary conditions.

In the case of (4), the boundary conditions yield

$$0 = F(0) = M,$$

and further

$$0 = F(L) = N\sin(\sqrt{-k}L) \Rightarrow \sqrt{-k} = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

Thus we can write

$$F_n(x) = N_n \sin(\frac{n\pi x}{L}), \quad n = 1, 2, 3, \ldots$$

Neumann boundary conditions.

In the case of (5), the boundary conditions yield

$$0 = F'(0) = \sqrt{-k}N,$$

which implies $N = 0$, and

$$0 = F'(L) = \sqrt{-k}M\sin\sqrt{-k}L,$$

which implies $\sqrt{-k} = n\pi/L$ for $n = 1, 2, 3, \ldots$.

Thus we can write

$$F_n(x) = M_n \cos\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Observation.

In both cases we have

$$k = -\frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, 3, \ldots$$

The function G.

With our new knowledge of k being either 0 (Neumann) or $-n^2\pi^2/L^2$ (Dirichlet and Neumann), the solutions of 3 are (again by what we know of ODEs from earlier courses)

$$G_n(t) = J_n\cos\frac{n\pi ct}{L} + K_n\sin\frac{n\pi ct}{L}$$

and

$$G_0(t) = J_0 t + K_0.$$

Every non-trivial separable solution of the wave equation is thus of the form

$$u_n(x,t) = F_n(x)G_n(t) = \left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L} \qquad \text{(Dirichlet conditions)}, \tag{6}$$

and

$$u_n(x,t) = F_n(x)G_n(t) = \left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right)\cos\frac{n\pi x}{L} \qquad \text{(Neumann conditions)}, \tag{7}$$

for $n = 1, 2, \ldots$ and constants $A_n = M_n J_n$ and $B_n = M_n K_n$, or

$$u_0(x,t) = F_0(x)G_0(t) = A_0 + B_0 t \qquad \text{(Neumann conditions)} \tag{8}$$

for constants $A_0 = AJ_0$ and $B_0 = AK_0$.

Throwback to linear algebra.

Since all u_n, $n = 0, 1, 2, 3, \ldots$ solves the wave equation (1), any linear combination will also solve (1). We show this by defining

$$u(x,t) = \sum_{n=0}^{N} c_n u_n(x,t),$$

for arbitrary constants $c_n \in \mathbb{R}$ and $N \in \mathbb{N}$. The boundary conditions are trivially satisfied for this linear combination (check this!). Furthermore,

$$\begin{aligned} u_{tt}(x,t) &= \partial_t\Big(\partial_t\Big(\sum_{n=0}^{N} c_n u_n(x,t)\Big)\Big) = \sum_{n=0}^{N} c_n \partial_t \partial_t u_n(x,t) \\ &= \sum_{n=0}^{N} c_n c \partial_x \partial_x u_n(x,t) = c\partial_x\Big(\partial_x\Big(\sum_{n=0}^{N} c_n u_n(x,t)\Big)\Big) = c u_{xx}(x,t), \end{aligned}$$

where we have used that u_n solves (1) and linearity of differentiation. We bravely conclude that any solution of (1) is given by

$$u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t) \tag{9}$$

where we have let $N \to \infty$. Here u_n is given by (6) (Dirichlet), or (7) and (8) (Neumann).

Observation.

Recall from linear algebra that whenever $y, z \in \mathbb{R}^n$ solves the equation $Ax = 0$, for some given matrix $A \in \mathbb{R}^{m\times n}$, then any linear combination of them, say $ay + bz$ also solves the equation:

$$A(ay + bz) = aAy + bAz = 0.$$

In linear algebra the solutions of the equation $Ax = 0$ defines a subspace of $\mathbb{R}^n$, $V := \{v \in \mathbb{R}^n \mid Av = 0\}$. If this subspace has dimension k, then it has a basis of linearly independent vectors $\{b_1, \ldots, b_k\}$ such that all $v \in V$ can be expressed as a linear combination of $b_1, \cdots, b_k$. As we have shown, we see similar behaviour for solutions of the wave equation (1).

Actually solving problem 3

We put $L = 3$ and $c = 1$ in (6) and obtain through (9) and (6)

$$u(x,t) = \sum_{n=1}^{\infty} F_n(x)G_n(t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi t}{3} + B_n \sin\frac{n\pi t}{3} \right) \sin\frac{n\pi x}{3}. \tag{10}$$

$\boxed{4}$ We use the general solution (10).

a) The first initial condition is

$$x(3-x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{3},$$

which shows that A_n is the n'th Fourier sine coefficient of (the odd periodic extension of) the function f given by $f(x) = x(3-x)$, i.e.

$$\begin{aligned}
A_n &= \frac{2}{3}\int_0^3 x(3-x)\sin\frac{n\pi x}{3}\,\mathrm{d}x \\
&= \frac{2}{3}\underbrace{\left[-\frac{3}{n\pi}x(3-x)\cos\frac{n\pi x}{3}\right]_0^3}_{=0} + \frac{2}{n\pi}\int_0^3 (3-2x)\cos\frac{n\pi x}{3}\,\mathrm{d}x \\
&= \frac{2}{n\pi}\underbrace{\left[\frac{3}{n\pi}(3-2x)\sin\frac{n\pi x}{3}\right]_0^3}_{=0} + \frac{12}{n^2\pi^2}\int_0^3 \sin\frac{n\pi x}{3}\,\mathrm{d}x \\
&= \frac{36}{n^3\pi^3}\left[-\cos\frac{n\pi x}{3}\right]_0^3 = \frac{36}{n^3\pi^3}\left(1-(-1)^n\right) = \begin{cases} \frac{72}{n^3\pi^3} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}
\end{aligned}$$

The exact same reasoning on the other initial condition gives that the B_n's must be the Fourier sine coefficients for the zero function as

$$0 = \frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi}{3} B_n \sin\frac{n\pi x}{3} \quad \Rightarrow \quad \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{3} = 0.$$

Thus, $B_n = 0$ for all n. The general solution is therefore

$$u(x,t) = \frac{72}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\cos\frac{(2n-1)\pi t}{3}\sin\frac{(2n-1)\pi x}{3}.$$

b) The same reasoning as in **a)** gives that now the A_n's are Fourier sine coefficients for the zero function, i.e. $0 = A_1 = A_2 = \dots$. The second initial condition yields

$$\sin \pi x - \frac{1}{2} \sin 2\pi x = \frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi}{3} B_n \sin \frac{n\pi x}{3}$$
$$\Rightarrow \sum_{n=1}^{\infty} B_n^* \sin \frac{n\pi x}{3} = \sin \pi x - \frac{1}{2} \sin 2\pi x,$$

with $B_n^* = \frac{n\pi}{3} B_n$. Hence, B_n^* are the Fourier sine coefficients of the function $f(x) = \sin \pi x - \frac{1}{2} \sin 2\pi x$. As this function can be written on Fourier sine series with period $L = 3$

$$f(x) = \sin \frac{3\pi x}{3} - \frac{1}{2} \sin \frac{6\pi x}{3}$$

we can extract the coefficients directly

$$B_n = \frac{3}{n\pi} B_n^* = \begin{cases} \frac{1}{\pi} & n = 3 \\ -\frac{1}{4\pi} & n = 6 \\ 0 & n \neq 3, 6. \end{cases}$$

Thus,

$$u(x,t) = \frac{1}{\pi} \sin(\pi t) \sin(\pi x) - \frac{1}{4\pi} \sin(2\pi t) \sin(2\pi x).$$

c) Let $u(x,t) = u_{\mathrm{a}}(x,t) + u_{\mathrm{b}}(x,t)$ be the sum of the solutions in part a and b. Then

$$u(x,0) = u_{\mathrm{a}}(x,0) + \underbrace{u_{\mathrm{b}}(x,0)}_{=0} = x(3-x)$$

and

$$\frac{\partial u(x,0)}{\partial t} = \underbrace{\frac{\partial u_{\mathrm{a}}(x,0)}{\partial t}}_{=0} + \frac{\partial u_{\mathrm{b}}(x,0)}{\partial t} = \sin \pi x - \frac{1}{2} \sin 2\pi x.$$

Due to uniqueness, this is thus the solution satisfying the given boundary conditions

$$u(x,t) = \frac{72}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos \frac{(2n-1)\pi t}{3} \sin \frac{(2n-1)\pi x}{3}$$
$$+ \frac{1}{\pi} \sin(\pi t) \sin(\pi x) - \frac{1}{4\pi} \sin(2\pi t) \sin(2\pi x).$$

Anbefalte oppgaver

$\boxed{1}$ See the lecture notes

$\boxed{2}$ The general solution is given by (9) with u_n given by (7) and (8).

The allowed frequencies for the flute is given by sinusoidal time-varying part of the solution of the wave equation, that is

$$\omega_n = \frac{n\pi c}{L}, \quad n = 0, 1, 2, \dots .$$

With the speed of sound as in the exercise text ($c = 343$ m/s), a flute of length 0.5 m allows for the following frequencies

$$\omega_1 \approx 2155\,\mathrm{s}^{-1}, \omega_2 \approx 4309\,\mathrm{s}^{-1}, \omega_3 \approx 6464\,\mathrm{s}^{-1}, \ldots,$$

so the deepest frequency (its fundamental mode) is about $2136\,\mathrm{s}^{-1}$ (the unit is "seconds-inverse", also known as Hertz).

The frequencies listed here are so-called *angular frequencies*. Typically, the frequency is the quantity of interest, and is defined by $f_n = \omega_n/(2\pi) = nc/(2L)$. The fundamental frequency mode is then $f_1 = 343\,\mathrm{Hz}$.