Øving 7 - Varmelikningen - LF

Obligatoriske oppgaver

1 See the lecture notes.

2 **Matlab:**

```
%se oving_5.m for kommentarer
x=0:.1:2;
t=0:pi/100:pi;

figure

for j=1:length(t)
    f=exp(-pi^2*t(j)/4)*sin(pi*x/2);
    plot(x,f)
    axis([0 2 -1.2 1.2])
    pause(.03)
end
```

Python:

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation

#samme kommoentar som oving 5 og 6
fig, ax = plt.subplots()
ax.set_xlabel(r'$x$')
ax.set_xlabel(r'$y$')

xdata, ydata = [], []
ln, = plt.plot([], [], animated=True,label=r'$u\, (x,t)$') # skriv inn 'ro' ...
    som tredje argument hvis du vil ha tilbake punkter i stedet for linjer.

def init():
    ax.set_xlim(0, 2)
    ax.set_ylim(-.5, 1.5)
    return ln,

def update(frame):
    xdata=np.linspace(0, 2, 128)
    ydata=np.exp(-np.pi**2*frame/4.0)*np.sin(np.pi*xdata/2)
    ln.set_data(xdata, ydata)
    return ln,

# Man kan endre hastigheten ved aa endre paa interval-parameteren
ani = FuncAnimation(fig, update, frames=np.linspace(0, 2, 128),interval=30,
                    init_func=init, blit=True)
plt.legend()
plt.show()
```

$\boxed{3}$ Separation of variables as given in the text turns the given (heat) equation into

$$F(x)G'(t) = F''(x)G(t).$$

Rearranging this equation gives

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{G(t)}.$$

Since this holds for all x and t, both sides must be equal to a constant k, so we have the two equations

$$F'' - kF = 0 \tag{1}$$
$$G' - kG = 0. \tag{2}$$

As the boundary conditions hold for all t, we have (by inserting $u(x,t) = F(x)G(t)$ into the boundary conditions)

$$F(0) = 0 \quad \text{and} \quad F(2) = 0 \tag{3}$$

Let us first study (4). From earlier courses we know that there are three types of solutions, depending on whether k is positive, negative or zero:

1. If $k = 0$, then $F(x) = A + Bx$ for constants A and B. For the boundary conditions in (6) to be satisfied, we must have $A = B = 0$, i.e. the trivial solution, which is of no interest.
2. If $k > 0$, then
$$F(x) = Ce^{\sqrt{k}x} + De^{-\sqrt{k}x}.$$
Since
$$0 = F(0) = C + D, \quad \text{and} \quad 0 = F(2) = Ce^{2\sqrt{k}} + De^{-2\sqrt{k}}$$
we get $C = D = 0$, and thus the trivial solution once more.
3. If $k < 0$, then
$$F(x) = M\cos(\sqrt{-k}x) + N\sin(\sqrt{-k}x).$$
Write $\mu = \sqrt{-k}$ to simplify notation from here on. The boundary conditions then become
$$0 = F(0) = M$$
and (since $M = 0$)
$$0 = F(2) = N\sin 2\mu.$$
Hence, $\mu = n\pi/2$ for $n \in \mathbb{Z}$.

Solutions of (4) are thus of the form

$$F_n(x) = N_n \sin\frac{n\pi x}{2}, \quad n = 1, 2, \ldots$$

where we can restrict ourselves to $n \geq 1$ as sine is antisymmetric.

With $k = -\mu^2$, (5) has the solution (Calculus 1 curriculum as the ODE is separable)

$$G_n(t) = J_n \mathrm{e}^{kt} = J_n \mathrm{e}^{-\pi^2 n^2 t/4}.$$

Every non-trivial separable solution of the wave equation is thus of the form

$$u_n(x,t) = F_n(x)G_n(t) = B_n \mathrm{e}^{-\pi^2 n^2 t/4} \sin\frac{n\pi x}{2}$$

for $n = 1, 2, \ldots$ and constants $B_n = N_n J_n$. By superposition, the general solution is therefore

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-\pi^2 n^2 t/4} \sin \frac{n\pi x}{2}$$

as

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2}, \quad \text{with} \quad f(x) = \begin{cases} x & 0 \le x \le 1 \\ -x+2 & 1 \le x \le 2 \end{cases}$$

the coefficients B_n will be the Fourier sine coefficients of (the odd extension of) $f(x)$. The coefficients are thus found by

$$\begin{aligned}
B_n &= \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} \, \mathrm{d}x \\
&= \int_0^1 x \sin \frac{n\pi x}{2} \, \mathrm{d}x + \int_1^2 (2-x) \sin \frac{n\pi x}{2} \, \mathrm{d}x \\
&= \left[-\frac{2x}{n\pi} \cos \frac{n\pi x}{2} \right]_0^1 + \frac{2}{n\pi} \int_0^1 \cos \frac{n\pi x}{2} \, \mathrm{d}x \\
&\quad + \left[-\frac{2(2-x)}{n\pi} \cos \frac{n\pi x}{2} \right]_1^2 - \frac{2}{n\pi} \int_1^2 \cos \frac{n\pi x}{2} \, \mathrm{d}x \\
&= -\frac{2}{n\pi} \cos \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \left[\sin \frac{n\pi x}{2} \right]_0^1 + \frac{2}{n\pi} \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \left[\sin \frac{n\pi x}{2} \right]_1^2 \\
&= \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} = \begin{cases} \frac{8(-1)^{\frac{n-1}{2}}}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}
\end{aligned}$$

Thus,

$$u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \mathrm{e}^{-\pi^2 (2n-1)^2 t/4} \sin \frac{(2n-1)\pi x}{2}.$$

$\boxed{4}$ The steady state solution $u(x,y)$ satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Using separation of variables of the form $u(x,y) = F(x)G(y)$, the Laplace equation turns into

$$F(x)G''(y) = -F''(x)G(y).$$

Rearranging this equation gives

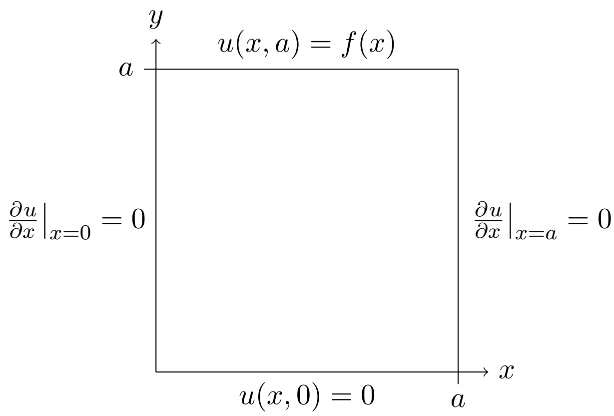
$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}.$$

Since this holds for all x and y, both sides must be equal to a constant k, so we have the two equations

$$F'' - kF = 0 \tag{4}$$
$$G'' + kG = 0. \tag{5}$$

As the boundary conditions (perfectly insulation, see Figure 1)

$$\left. \frac{\partial u}{\partial x} \right|_{x=0} = 0 \quad \text{and} \quad \left. \frac{\partial u}{\partial x} \right|_{x=a} = 0$$



Figur 1: **Task 3**: Domain of the Laplace equation with boundary conditions.

holds for all y, we have (by inserting $u(x,y) = F(x)G(y)$ into the boundary conditions)

$$F'(0) = 0 \quad \text{and} \quad F'(a) = 0 \tag{6}$$

As before, we start by solving (4) subject to the boundary conditions (6). From earlier courses we know that there are three types of solutions, depending on whether k is positive, negative or zero:

1. If $k = 0$, then $F(x) = A + Bx$ for constants A and B. For the boundary conditions in (6) to be satisfied, we must have $B = 0$, i.e. we have the constant solution $F(x) = A$.
2. If $k > 0$, then
$$F(x) = Ce^{\sqrt{k}x} + De^{-\sqrt{k}x},$$
and so
$$F'(x) = C\sqrt{k}e^{\sqrt{k}x} - D\sqrt{k}e^{-\sqrt{k}x}.$$
Since
$$0 = F'(0) = C\sqrt{k} - D\sqrt{k}, \quad \text{and} \quad 0 = G'(a) = C\sqrt{k}e^{a\sqrt{k}} - D\sqrt{k}e^{-a\sqrt{k}}$$
we get $C = D = 0$, and thus the trivial solution which is of no interest.
3. If $k < 0$, then
$$F(x) = M\cos(\sqrt{-k}x) + N\sin(\sqrt{-k}x).$$
and so
$$F'(x) = -M\sqrt{-k}\sin(\sqrt{-k}x) + N\sqrt{-k}\cos(\sqrt{-k}x).$$
Write $\mu = \sqrt{-k}$ to simplify notation from here on. The boundary conditions then imply
$$0 = G'(0) = N\sqrt{-k}$$
and (since $N = 0$)
$$0 = F'(a) = -M\sqrt{-k}\sin a\mu.$$
Hence, $\mu = n\pi/a$ for $n \in \mathbb{Z}$.

Solutions of (4) are thus of the form

$$F_n(x) = M_n \cos\frac{n\pi x}{a}, \quad n = 0, 1, 2, \dots$$

where we can restrict ourselves to $n \geq 0$ as cosine is symmetric.

As the boundary conditions (see Figure 1) $u(x, 0) = 0$ holds for all x, we have (by inserting $u(x, y) = F(x)G(y)$ into the boundary condition)

$$G(0) = 0 \tag{7}$$

If $k = 0$ then (4) has solution $G_0(y) = J_0 + K_0 y$, and the boundary condition in (7) implies that $J_0 = 0$. That is, we have the linear solution $G_0(y) = K_0 y$.

With $k = -\mu^2$, (4) has the solution (again curriculum from previous courses)

$$G_n(y) = J_n \sinh \mu y + K_n \cosh \mu y.$$

The boundary condition (7) yields

$$0 = G_n(0) = K_n$$

Every non-trivial separable solution of the wave equation is thus of the form

$$u_n(x, y) = F_n(x)G_n(y) = A_n \cos \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

for $n = 1, 2, \ldots$ and constants $A_n = M_n J_n$, or $u_0(x, y) = A_0 y$ (with $A_0 = AK_0$). By superposition, the general solution is therefore

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

As

$$f(x) = u(x, a) = A_0^* + \sum_{n=1}^{\infty} A_n^* \cos \frac{n\pi x}{a}$$

where

$$f(x) = \cos \frac{\pi x}{6} = \cos \frac{4\pi x}{a} \quad \text{and} \quad A_n^* = \begin{cases} \sinh(n\pi) A_n & n \geq 1 \\ A_0 a & n = 0 \end{cases}$$

the coefficients A_n^* the coefficients can be extracted as

$$A_n^* = \begin{cases} 1 & n = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$A_n = \begin{cases} \frac{1}{\sinh n\pi} & n = 4 \\ 0 & \text{otherwise.} \end{cases}$$

The final solution is therefore (with $a = 24$)

$$u(x, y) = \frac{1}{\sinh 4\pi} \cos \frac{\pi x}{6} \sinh \frac{\pi y}{6}.$$

$\boxed{5}$ **Python:**

```
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter
import numpy as np


#x- og y-aksen
x = np.linspace(0, 2, 128)
t = np.linspace(0, 2, 128)

#ytreprodukt av x- og y-aksen. surf trenger dette for aa funke.
X,T=np.meshgrid(x,t)

#funksjonsverdiene
u=np.multiply(np.exp(-np.pi**2*T/4.0),np.sin(np.pi*X/2))

#lage plot. et par av disse kommandoene vet jeg ikke hva gjor, men jeg fant ...
    dem paa nettet
fig = plt.figure()

ax = fig.gca(projection='3d')
surf = ax.plot_surface(X, T, u, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False)

#korrekt utsnitt av xy-planet
plt.axis([0,2,0,2])

# navn paa aksene
plt.xlabel(r'$x$')
plt.ylabel(r'$t$')
ax.set_zlabel(r'$u(x,t)$')

#vise plot
plt.show()
```

Anbefalte oppgaver

$\boxed{1}$ We want to solve the Schrödingers equation given by

$$\begin{cases} u_t = iu_{xx} \text{ for } t>0, -\infty < x < +\infty \\ u(x,0) = g(x) \text{ on } -\infty < x < +\infty. \end{cases} \tag{8}$$

Note that this equation is on the form of the heat equation

$$\begin{cases} u_t = c^2 u_{xx} \text{ for } t>0, -\infty < x < +\infty \\ u(x,0) = g(x) \text{ on } -\infty < x < +\infty. \end{cases} \tag{9}$$

where $c^2 = i$. We have $c = e^{\frac{\pi i}{4}} \vee e^{\frac{5\pi i}{4}}$ by complex analysis. Recall the solution formula for (9)

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\nu) e^{-\frac{(x-\nu)^2}{4c^2 t}} \, d\nu.$$

We get

$$u_1(x,t) = \frac{1}{2e^{\frac{\pi i}{4}}\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\nu) e^{-\frac{(x-\nu)^2}{4it}} d\nu,$$

and

$$u_2(x,t) = \frac{1}{2e^{\frac{5\pi i}{4}}\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\nu) e^{-\frac{(x-\nu)^2}{4it}} d\nu.$$

Note that $u_1 = e^{\pi i} u_2$, so we can throw away the solution u_2 and describe everything in terms of u_1. Finally, the solution of (8) is given by

$$u(x,t) = \frac{1}{2e^{\frac{\pi i}{4}}\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\nu) e^{-\frac{(x-\nu)^2}{4it}} d\nu.$$

$\boxed{2}$ **a)** We are going to take the Fourier transform with respect to the x-variable, in the end transforming the PDE into an ODE (easier to solve!). First, we recall the relation

$$\mathscr{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -w^2\,\mathscr{F}\{u\} = -w^2 \hat{u}.$$

Moreover, assuming we may interchange the order of differentiation and integration, we have

$$\mathscr{F}\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2} \mathrm{e}^{-\mathrm{i}xw}\,\mathrm{d}x = \frac{\partial^2}{\partial y^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) \mathrm{e}^{-\mathrm{i}xw}\,\mathrm{d}x = \frac{\partial^2 \hat{u}}{\partial y^2}.$$

Thus, using the linearity of Fourier transform , taking the Fourier transform in x of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

yields the ODE

$$\frac{\partial^2 \hat{u}}{\partial y^2} - w^2 \hat{u} = 0. \tag{10}$$

b) The *ordinary* differential equation in (10) has the characteristic polynomial $\lambda^2 - w^2 = 0$ with solutions $\lambda = \pm|w|$. Recall from theory of linear ordinary differential equations that we then have solutions of the form

$$\hat{u}(w,y) = C(w)\mathrm{e}^{-|w|y} + D(w)\mathrm{e}^{|w|y} \tag{11}$$

Assuming we can interchange the order of the limit and integration, we have (using the last boundary condition)

$$\begin{aligned}
\lim_{y\to\infty} \hat{u}(w,y) &= \lim_{y\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y)\mathrm{e}^{-\mathrm{i}xw}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\lim_{y\to\infty} u(x,y)}_{=0}\, \mathrm{e}^{-\mathrm{i}xw}\,\mathrm{d}x = 0
\end{aligned}$$

For this reason, we must have $D(w) = 0$ in (11). The solution of (10) is therefore

$$\hat{u}(w,y) = C(w)\mathrm{e}^{-|w|y} \tag{12}$$

for some function $C(w)$.

c) Since

$$u(x,0) = f(x) \quad \Rightarrow \quad \hat{u}(w,0) = \hat{f}(w)$$

we get

$$\hat{f}(w) = \hat{u}(w,0) = C(w) \cdot \mathrm{e}^{-|w| \cdot 0} = C(w).$$

That is,

$$C(w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}wx}\,\mathrm{d}x.$$

d) Applying the inverse Fourier transform (w.r.t. w) of both sides of (12) yields

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\mathrm{e}^{-|w|y}\mathrm{e}^{\mathrm{i}wx}\,\mathrm{d}w.$$

e) Recall the Fourier transform of a convolution

$$\mathscr{F}\{f * g\}(w) = \sqrt{2\pi}\hat{f}(w)\hat{g}(w) \quad \text{where} \quad (f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,\mathrm{d}t.$$

Using $\hat{g}(w) = \mathrm{e}^{-|w|y}$ and the fact that

$$\mathscr{F}\left\{\frac{1}{x^2+a^2}\right\} \overset{a \geq 0}{=} \sqrt{\frac{\pi}{2}}\frac{e^{-a|\omega|}}{a},$$

we get

$$\mathscr{F}^{-1}\left\{\sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{-|w|y}}{y}\right\} = \mathscr{F}^{-1}\left\{\sqrt{\frac{\pi}{2}}\frac{1}{y}\hat{g}(w)\right\} = \frac{1}{x^2+y^2}$$
$$\Rightarrow \quad g(x) = \sqrt{\frac{2}{\pi}}\frac{y}{x^2+y^2}.$$

So further

$$(f * g)(x) = \mathscr{F}^{-1}\left\{\sqrt{2\pi}\hat{f}(w)\hat{g}(w)\right\} = \int_{-\infty}^{\infty} \hat{f}(w)\mathrm{e}^{-|w|y}\mathrm{e}^{\mathrm{i}wx}\,\mathrm{d}w = \sqrt{2\pi}u(x,y),$$

and

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,\mathrm{d}t = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(t)\frac{y}{(x-t)^2+y^2}\,\mathrm{d}t.$$

That is,

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\frac{y}{(x-t)^2+y^2}\,\mathrm{d}t.$$