Øving 9 - Interpolasjon - LF

Obligatoriske oppgaver

$\boxed{1}$ We start by calculating the following Taylor expansions around x:

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + O(h^4),$$

and

$$f(x-2h) = f(x) - f'(x)\cdot 2h + \frac{f''(x)}{2}4h^2 - \frac{f'''(x)}{6}8h^3 + O(h^4).$$

For constants $a, b, c \in \mathbb{R}$, we write a linear combination of $f(x), f(x-h)$ and $f(x-2h)$ as follows

$$\begin{aligned} af(x) + bf(x-h) + cf(x-2h) = af(x) &+ b\Big[f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + O(h^4)\Big] \\ &+ c\Big[f(x) - f'(x)\cdot 2h + \frac{f''(x)}{2}4h^2 - \frac{f'''(x)}{6}8h^3 + O(h^4)\Big]. \end{aligned}$$

We sort the terms by $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$.

$$(a+b+c)f(x) + (-b-2c)f'(x)h + (b+4c)\frac{f''(x)}{2}h^2 + (-b-8c)\frac{f'''(x)}{6}h^3 + O(h^4).$$

For the best convergence rate we want the following equations to hold

$$\begin{aligned} a+b+c &= 0, \\ b+4c &= 0. \end{aligned}$$

This yields

$$b = -4c$$

and then

$$a - 3c = 0 \quad \Rightarrow \quad a = 3c.$$

Inserting this into our calculations we get

$$\begin{aligned} 3cf(x) - 4cf(x-h) + cf(x-2h) &= 0\cdot f(x) + 2cf'(x)h + 0f''(x)h^2 - 4c\frac{f'''(x)}{6}h^3 + O(h^4) \\ &= 2cf'(x)h + O(h^3). \end{aligned}$$

For simplicity we choose $c = 1/2$. Dividing both sides by h we get

$$\frac{3f(x) - 4f(x-h) + f(x-2h)}{2h^2} = f'(x) + O(h^2),$$

which is what we wanted. The rate is quadratic.

n	x_n	f_n
0	-2	-2
1	1	1
2	0	2
3	1	-1

Tabell 1: Interpolation points for exercise 4.

$\boxed{2}$ We calculate the approximate derivatives, and the corresponding errors. We know that $f'(1) = \cos(1)$, so we can compute the errors directly in the calculator using the formula $e_h = |\text{approximation} - \cos(1)|$.

$$h = 0.1 \Rightarrow \quad \frac{\sin(1+0.1) - \sin(1-0.1)}{2 \cdot 0.1} \approx 0.5394022522, \qquad e_h = 9 \cdot 10^{-4}$$
$$h = 0.01 \Rightarrow \quad \frac{\sin(1+0.01) - \sin(1-0.01)}{2 \cdot 0.01} \approx 0.5402933009, \qquad e_h = 9 \cdot 10^{-6}$$
$$h = 0.001 \Rightarrow \quad \frac{\sin(1+0.001) - \sin(1-0.001)}{2 \cdot 0.001} \approx 0.5403022158, \qquad e_h = 9 \cdot 10^{-8}$$

What we see here is quadratic convergence. **Why?** Assume that the method has convergence rate p, that is

$$e_h \leq Ch^p,$$

for some constant $C > 0$. If we insert $0.1h$ into this relation we get

$$e_{0.1h} = C(0.1h)^p = 0.1^p Ch^p = 0.1^p e_h$$

In our case we see that

$$\frac{e_{0.001}}{e_{0.01}} = \frac{e_{0.01}}{e_{0.1}} = 0.01 = 0.1^2,$$

so that $p = 2$, hence quadratic convergence. A step length of $h = 0.0001$ would then give an error of size $9 \cdot 10^{-10}$, which is sufficiently close for all practical purposes.

$\boxed{3}$ See the lecture notes.

$\boxed{4}$ The interpolation points for an equidistant grid are given in Table 1. We calculate the following Lagrange polynomials

$$L_0(x) = \frac{l_0(x)}{l_0(x_0)} = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = -\frac{x}{6}\left(x^2-1\right)$$
$$L_1(x) = \frac{l_1(x)}{l_1(x_0)} = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{x}{2}(x-1)(x+2)$$
$$L_2(x) = \frac{l_2(x)}{l_2(x_0)} = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = -\frac{1}{2}(x+2)\left(x^2-1\right)$$
$$L_3(x) = \frac{l_3(x)}{l_3(x_0)} = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{x}{6}(x+1)(x+2)$$

such that the interpolating polynomial is given by

$$p_3(x) = \sum_{k=0}^{3} L_k(x) f_k = -2L_0(x) + L_1(x) + 2L_2(x) - L_3(x) = -\frac{1}{3}x^3 - 2x^2 - \frac{2}{3}x + 2. \quad (1)$$

x_k	f_k	$f[x_k, x_{k+1}]$	$f[x_k, x_{k+1}, x_{k+2}]$	$f[x_k, x_{k+1}, x_{k+2}, x_{k+3}]$
-2	-2			
		3		
-1	1		-1	
		1		$-\frac{1}{3}$
0	2		-2	
		-3		
1	-1			

Tabell 2: Newton divided differences table used in exercise 4.

Alternatively: The polynomial can also be found using Newton's divided differences. The coefficients

$$a_k = f[x_0, \dots, x_k] = \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}$$

can be computed as in Table 2. The highlighted numbers in blue are the coefficients of the Newton polynomial

$$\begin{aligned} p_3(x) &= -2 + 3(x+2) - 1(x+2)(x+1) - \frac{1}{3}(x+2)(x+1)x \\ &= -\frac{1}{3}x^3 - 2x^2 - \frac{2}{3}x + 2. \end{aligned}$$

$\boxed{5}$ The Chebyshev point on a interval $[a, b] = [-1, 2]$ is given by

$$\tilde{x}_k = \frac{(b-a)x_k + a + b}{2} = \frac{b-a}{2}\cos\left(\frac{2k+1}{2n+2}\pi\right) + \frac{a+b}{2} = \frac{3}{2}\cos\left(\frac{2k+1}{2n+2}\pi\right) + \frac{1}{2}$$

for $k = 0, \dots, n$ with $n = 3$. Approximate values for these four points are given by

$$\tilde{x}_0 \approx 1.886, \quad \tilde{x}_1 \approx 1.074, \quad \tilde{x}_2 \approx -0.074, \quad \tilde{x}_3 \approx -0.886.$$

The coefficients of the interpolating polynomials at these points, can again be computed by Newton's divided differences. Using the code below we obtain the polynomial

$$p_3(x) = a_0 + a_1(x - \tilde{x}_0) + a_2(x - \tilde{x}_0)(x - \tilde{x}_1) + a_3(x - \tilde{x}_0)(x - \tilde{x}_1)(x - \tilde{x}_2)$$

with the following approximation of the coefficients

$$a_0 \approx -1.102, \quad a_1 \approx -2.035, \quad a_2 \approx -1.280, \quad a_3 \approx -0.660.$$

Python:

```
import numpy as np

# Create function to find coefficients for Newton divided differences
def newtonDivDiff(X,Y):
    m = len(X)  # number of interpolation points
    for j in range(1,m):
        for i in range(m-1,j-1,-1):
            Y[i] = (Y[i] - Y[i-1])/(X[i] - X[i-j])
    return Y
#Chebychev points
X = np.array([1.886, 1.074, -0.074, -0.886])
```

```
#Corresponding function values
Y = np.multiply(np.multiply(X,X), np.cos(X))

# Answer to 6
Y = newtonDivDiff(X,Y)
print('The Newton coefficients for p_3 are:', Y)
```

Anbefalte oppgaver

$\boxed{1}$ See the lecture notes.

$\boxed{2}$ For convenience we calculate the following Taylor expansions

$$\begin{aligned} f(x+2h) &= f(x) + f'(x)\cdot 2h + \frac{f''(x)}{2}4h^2 + \frac{f'''(x)}{6}8h^3 + O(h^4), \\ f(x-2h) &= f(x) - f'(x)\cdot 2h + \frac{f''(x)}{2}4h^2 - \frac{f'''(x)}{6}8h^3 + O(h^4), \end{aligned}$$

and

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3 + O(h^4), \\ f(x-h) &= f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + O(h^4). \end{aligned}$$

We immediately get cancellations when calculating the following terms:

$$f(x+2h) + f(x-2h) = 2f(x) + 8\frac{f''(x)}{2}h^2 + O(h^4) \tag{2}$$

and

$$f(x+h) + f(x-h) = 2f(x) + 2\frac{f''(x)}{2}h^2 + O(h^4). \tag{3}$$

We calculate our numerator by first carefully rearranging its terms, and then using (2) and (3). We get

$$\begin{aligned} &-f(x-2h) + 16f(x-h) - 30f(x) + 16f(x+h) - f(x+2h) \\ &= -(f(x+2h) + f(x-2h)) + 16(f(x+h) + f(x-h)) - 30f(x) \\ &= -(2f(x) + 8\frac{f''(x)}{2}h^2 + O(h^4)) + 16(2f(x) + 2\frac{f''(x)}{2}h^2 + O(h^4)) - 30f(x) \\ &= \underbrace{\cdots}_{\text{check this!}} = 12f''(x)h^2 + O(h^4). \end{aligned}$$

Dividing this expression by $12h^2$ we obtain

$$\frac{-f(x-2h) + 16f(x-h) - 30f(x) + 16f(x+h) - f(x+2h)}{12h^2} = f''(x) + O(h^2),$$

which means that we have a second order method of computing the second derivative!

$\boxed{3}$ We use the estimates of $(\sin(x))'_{x=1}$ that we found in exercice 3. We define the following function

$$\phi(h) = \frac{\sin(1+h) - \sin(1-h)}{2h}$$

For an approximation of order n, the Richardson extrapolation formula is given by (see Wikipedia),

$$R(h,t) = \frac{t^n \phi(h/t) - \phi(h)}{t^n - 1}$$

where h is the step size and t is a scaling factor.

In our case: $n = 2$ and $t = 10$. Plugging in our estimates into this formula we get for $h = 0.1$ and $h = 0.01$

$$\begin{aligned} R(0.1, 10) &= \frac{10^2 \phi(0.01) - \phi(0.1)}{10^2 - 1} \\ &\approx \frac{100 \cdot 0.5402933009 - 0.5394022522}{99} \\ &\approx 0.5403023096. \end{aligned}$$

The error is now $4 \cdot 10^{-9}$. Further, for $h = 0.01$ and $h = 0.001$ we get

$$\begin{aligned} R(0.01, 10) &= \frac{10^2 \phi(0.001) - \phi(0.01)}{10^2 - 1} \\ &\approx \frac{100 \cdot 0.5403022158 - 0.5402933009}{99} \\ &\approx 0.5403023058. \end{aligned}$$

Here the error is smaller than $2 \cdot 10^{-11}$.

The morale of the story is that the error get even smaller by using this method.