Exercises 1

January 11, 2020

Mandatory

1

a) Without any explicit calculations, show that the function $f(t) = t^n$, with n being a positive whole number, has a Laplace transform.
b) Calculate the Laplace transforms of

$$f(t) = t$$
$$f(t) = t^2$$
$$f(t) = t^n, \quad n \text{ positive integer.}$$

2

Write a program (in Matlab or Python code) that plots the function $f(t) = e^{-\frac{1}{2}t^2}\cos(t)$ in the interval $(-\pi, \pi)$.

3

Use the shift-theorem for Laplace transforms to show that the Laplace transform of the function $f(t) = \cosh(t)\cos(t)$ is

$$F(s) = \frac{1}{2}\left(\frac{s-1}{(s-1)^2+1} + \frac{s+1}{(s+1)^2+1}\right).$$

4

Calculate the inverse Laplace transform of the functions
a)

$$F(s) = \frac{2s}{s^2 - 8s + 6}$$

b)

$$F(s) = \frac{s^3 - 2s + 4}{s^4 - 2s^3}.$$

5

Solve the initial-value problems

a)

$$y''(t) + 5y'(t) + 6y(t) = 0, \quad y(0) = -2, \quad y'(0) = 1$$

b)

$$y''(t) + 3y'(t) + 2y(t) = e^{-t}, \quad y(0) = y'(0) = 0.$$

Recommended exercises

6

Let $g(t) = f(ct)$, and let $\mathcal{L}(f) = F$ and $\mathcal{L}(g) = G$. Show that if $c > 0$, er

$$G(s) = \frac{1}{c} F\left(\frac{s}{c}\right)$$

7

Show that

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$$

8

A function f has period p if $f(t + p) = f(t)$ for all t. Show that the Laplace transform to such a function is given by

$$\mathcal{L}(f) = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t) dt.$$

9

$y'' - 2y' + 2y = 6e^{-t} \quad y(0) = 0, \quad y'(0) = 1$

10

Find the Laplace transforms for

a) $f(t) = \sinh(At)$

b) $f(t) = \cosh(At)$

c) $f(t) = \begin{cases} 0 & 0 < t < \pi \\ 1 & \text{else} \end{cases}$

c) $f(t) = \begin{cases} 0 & 0 < t < \pi \\ \cos t & \text{else} \end{cases}$

e) $f(t) = t^2 e^t$
f) $f(t) = e^t \cos t$
g) $f(t) = e^t \sin t$