Assignment 2: Laplace Transform II

January 20, 2020

Mandatory

1

Let f and F be functions with $F' = f$. Find the convolution $f \star g$, when g is the function

a

$g(t) = \delta(t-a), \quad a > 0.$

b

$g(t) = u(t-a), \quad a > 0.$

2

For $k > 0$, define the function

$$w_k(t) = \begin{cases} kt, & 0 \leq t \leq 1/k, \\ 1, & \text{else.} \end{cases}$$

a

Find the Laplace transform of w_k.

b

For $k > 0$, define the function

$$g_k(t) = \begin{cases} k, & 0 \leq t \leq 1/k, \\ 0, & \text{else.} \end{cases}$$

Use the Laplace transform to show that $y(t) = w_k(t)$ solves the initial-value problem

$$y'(t) = g_k(t), \quad y(0) = 0.$$

c

What functions do w_k and g_k converge to as $k \to \infty$?

3

Solve the initial-value problems

a

$y'' + 5y' + 6y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = -1.$

b

$y'(t) - \int_0^t (t-\tau)y(\tau)d\tau = t, \quad y(0) = 1.$

Recommended exercises

4

Derive the formulas for s-shift and t-shift.

5

Find the inverse Laplace transforms for

a

$$\frac{1}{s^2(s^2+1)}$$

b

$$\frac{s}{s^2+2s+1}$$

c

$$\frac{2s}{(s^2+1)^2}$$

d

$$(s-3)^{-5}$$

6

Find the Laplace transforms for

a

$$f(t) = (u(t) - u(t-\pi))\cos t$$

b

$$f(t) = u(t-a)t^2, \quad a > 0$$

c

$$f(t) = u(t) + 2\sum_{i=1}^{\infty}(-1)^i u(t - ia), \quad a > 0$$

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Solve the equations

a

$y'' + y = u(t - \pi) \quad y(0) = 0,\ y'(0) = 0$

b

$$y - y \star t = t$$

8

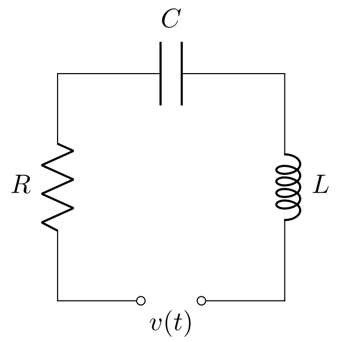
The current $i(t)$ in the circuit below is given by the equation below

$$Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t). \tag{1}$$

Assume $i(0) = i'(0) = 0$, $R = 4\ \Omega$, $L = 1$ H, $C = 0.05$ F and

$$v(t) = \begin{cases} 34e^{-t}\ \text{V} & \text{if 0¡t¡4,} \\ 0\ \text{V} & \text{else.} \end{cases}$$

Find the current i.



9

a

Detail the calculation showing that $\mathcal{L}(t^n)(s) = \frac{\Gamma(n+1)}{s^{n+1}}$, where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, $(x > 0)$, is the classical Gamma-function.

b

Show that $\Gamma(x + 1) = x\Gamma(x)$ and calculate $\Gamma(\frac{2k+1}{2})$, for k being a non-negative integer.

c

Show that $\Gamma(1/2) = 2\int_0^\infty e^{-p^2} dp$.

d

Prove that $2\int_0^\infty e^{-p^2} dp = \sqrt{\pi}$.

e

Combine part b), c) and d) to calculate the Laplace transform $\mathcal{L}(\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-p^2} dp)$.

Hint (at least three ways):

1. Expanding the exponential e^{-p^2} by using $e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots$;
2. Use definition of Laplace transform and two variables calculus;
3. Compute derivative of $\int_0^{\sqrt{t}} e^{-p^2} dp$ then use $\mathcal{L}(f')(s) = sF(s) - f(0)$.