Exercises 3

January 26, 2020

Mandatory

1

Find the trigonometric series $a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ to the function f when

a

$f(x) = x$,

b

$f(x) = |x|$,

c

$$f(x) = \begin{cases} \sin(x), & |x| \leq 2, \\ 0, & \text{else} \end{cases}.$$

2

Define $f_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$. Write a script that plots f_N for the series' you found in exercise 1**a-c**. Plot them on the interval $[-\pi, \pi]$, and use $N = 1, 3, 5$. Also plot the exact functions $f(x)$.

3

Define

$$w(t) = \begin{cases} 1, & 2k\pi <= x < (2k+1)\pi, \quad \text{for some } k \in \mathbb{N}, \\ -1, & \text{else.} \end{cases}$$

This is known as the square-wave.

a

Write w as an infinite series, where each term is a (shifted) scaled heavy-side function. That is,

$$w(t) = \sum_{k=0}^{\infty} c_k u(t - a_k).$$

b

Find the Laplace transform of w.

c

Use the Laplace transform to solve the equation

$$y'' + y = w, \quad y'(0) = y(0) = 0.$$

(Hint: you might need to use one of the shift theorems, and the Laplace transform of an integral.)

d

Plots $y(t)$ in the interval $[0, 10\pi]$. Note that you do not need to calculate all therms in the series. Also plot the function $t\sin(t)$.

4

a

Use Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ to show that

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

and

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a).$$

b

Show that

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \begin{cases} \pi, & n = m \\ 0, & n \neq m \end{cases}.$$

Recommended exercises

5

Find the Fourier series for the 2π-periodic function given by

$$f(x) = \begin{cases} -\sin x & \text{for} \quad -\pi \leq x \leq 0 \\ \sin x & \text{for} \quad 0 \leq x \leq \pi \end{cases}$$

and find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

6

Find the complex Fourier coefficients for the function

$$f(x) = 5 - 4\cos 2x - 2\sin 5x + 5\cos 8x.$$

7

Assume f is a smooth function with period 2π, with Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Find the Fourier series to $f(3x)$.