Exercises 4

February 2, 2020

Mandatory

1

A function f is called even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Show that if f is even, then

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

and if f is odd

$$\int_{-L}^{L} f(x)dx = 0$$

for any $L > 0$.

2

a

Show that the set of functions $\left\{1, x, \frac{3}{2}x^2 - \frac{1}{2}, \frac{5}{2}x^3 - \frac{3}{2}x\right\}$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.$$

b

Use the Gram-Schmidt orthogonalization process on the set of functions $\left\{1, x, x^2, x^3\right\}$ to get the set of orthogonal functions from the previous task. Rescale each function by requiring that $\phi_k(1) = 1$.

Hint: Given a set of functions $\{f_1, ..., f_n\}$ we can generate an orthogonal set of functions $\{\phi_1, ..., \phi_n\}$ by

$$\phi_k = f_k - \sum_{j=0}^{k-1} \frac{\langle f_k, \phi_j \rangle}{\|\phi_j\|^2} \phi_j$$

where $\|\phi_j\|^2 = \langle \phi_j, \phi_j \rangle$.

c

Find the polynomial p of degree ≤ 3 which minimizes the expression

$$\|p - f\|$$

when $f(x) = \cos(x)$.

3

a

Show that the set of functions $\left\{e^{i\frac{n\pi x}{L}}\right\}_{n=-\infty}^{\infty}$ is an orthogonal set with respect to the inner product

$$\langle f, g\rangle = \int_{-L}^{L} f(x)\overline{g(x)}dx.$$

b

Find the complex Fourier series of the function $f(x) = x + u(x)$, that is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}.$$

c

Write the Fourier series above as a trigonometric series, i.e.

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L).$$

Recommended

4

Sketch the function

$$f(x) = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < x < \pi \end{cases}$$

and its even and odd extensions. Find the Fourier series for the extensions.

5

Find the Fourier series to the function f with period one, where

$$f(x) = \cos \pi x \quad -\frac{1}{2} < x < \frac{1}{2}.$$

6

Use the Fourier series of x^2 on the interval $[-1, 1]$ to calculate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}.$$

Hint: You can integrate the series term by term.