Norwegian University of Science
and Technology
Department for Mathematical
Science

TMA4125 Calculus 4N
Spring 2020

Exercise set 5
Fourier transform

Mandatory exercises

1 Let $p(w) = \sum_{l=0}^{k} a_l w^l$ be a polynomial of order k. Then $p(\frac{\mathrm{d}}{\mathrm{d}x})f$ is given by

$$p(\tfrac{\mathrm{d}}{\mathrm{d}x})f(x) = \sum_{l=0}^{k} a_l \frac{\mathrm{d}^l}{\mathrm{d}x^l} f(x).$$

Assume that $f, f', \dots, f^k$ are all absolutely integrable on $\mathbb{R}$, and assume that $f(x), \dots, f^{k-1}(x) \to 0$ for $x \to \pm\infty$. Show that

$$\mathcal{F}(p(\tfrac{\mathrm{d}}{\mathrm{d}x})f) = p(iw)\mathcal{F}(f).$$

2 Compute the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$$

by completing the following outline:

Start from $\left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)^2$ and rewrite this expression as double integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \,\mathrm{d}x dy$. Then use polar cooordinates to compute the double integral.

3 Compute the convolution $f * f$ for $f(x) = e^{-x^2}$.

Hint: Don't try to compute it directly from the definition of the convolution, but rather exploit the convolution theorem for the Fourier transform.

4 For some positive constant $a > 0$, define the function

$$f(x) = \begin{cases} 1 & |x| \leqslant a, \\ 0 & \text{else,} \end{cases}$$

and show that its Fourier transform is

$$\hat{f}(w) = \sqrt{\frac{2}{\pi}}\frac{\sin aw}{w}.$$

5 Define the function

$$g(x) = \begin{cases} 1 - |x| & |x| \leqslant 1, \\ 0 & \text{else.} \end{cases}$$

Take the function f from Problem **4** with $a = 1/2$ and show that

$$f * f = g.$$

Use this to compute the Fourier transform $\mathcal{F}(g)$.

Recommended exercises

6 Compute the Fourier transform of $f(x) = e^{-|x|}$, and compute the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

7 Find the Fourier transform of $f(x) = x^2 e^{-x^2}$. (Hint: It might be clever to differentiate e^{-x^2} twice.)

8 Let $f(x) = e^{-x^2}$ and $g(x) = xe^{-x^2}$. Show that

$$f * g = -\frac{i}{4} \int_{-\infty}^{\infty} we^{-\frac{w^2}{2}} e^{iwx}\, dw.$$

9 Compute the Fourier transform for

$$f(x) = \begin{cases} \sin x & -\pi < x < \pi \\ 0 & \text{ellers} \end{cases}$$

and calculate the integral

$$\int_{-\infty}^{\infty} \frac{\sin(\pi w)\sin(\pi w/2)}{1-w^2}\, dw.$$