Exercises 6

February 17, 2020

Mandatory

1

You have seen in class that the solution to

$$\begin{cases} u_t(x,t) = c^2 u_{xx}(x,t), \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = f(x), \quad x \in \mathbb{R} \end{cases}$$

can be written as

$$u(x,t) = \left(f * \Phi^t\right)(x) = \int_{-\infty}^{\infty} f(v)\Phi^t(x-v)dx, \quad t > 0$$

where

$$\Phi^t(x) = \frac{1}{2c\sqrt{\pi t}} e^{-\frac{x^2}{4c^2 t}}, \quad t > 0.$$

a

For $t > 0$, calculate $\int_{-\infty}^{\infty} \Phi^t(x)dx$.

b

Plot $\Phi^t(x)$ in the interval $x \in [-1,1]$ for $t = 1, \frac{1}{10} \frac{1}{100}$. You can plot in whatever program you want (e.g. python, Matlab, Geogebra etc), and you only need to turn in plots, no code.

c

What is $\lim_{t \to 0} \Phi^t(x)$ if $x \neq 0$? What is it when $x = 0$?

d

To what "function" does Φ^t converge to as $t \to 0$?

2

We will now solve the heat equation with mixed boundary conditions.

a

Show that if m and n are positive whole numbers, then

$$\int_0^1 \cos\left(\left(n+\frac{1}{2}\right)\pi x\right)\cos\left(\left(m+\frac{1}{2}\right)\pi x\right)dx = \begin{cases} \frac{1}{2}, & m=n \\ 0, & m\neq n \end{cases} \quad .$$

Hint: Use product-to-sum formulas.

b

Suppose we know that a function f can be written as

$$f(x) = \sum_{n=0}^{\infty} b_n \cos\left(\left(n+\frac{1}{2}\right)\pi x\right).$$

Show that the coefficients b_n are given by

$$b_n = 2\int_0^1 f(x)\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)dx.$$

c

Solve the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & 0<x<1, \quad t>0 \\ u_x(0,t) = 0, & t>0 \\ u(1,t) = -1, & t>0 \\ u(x,0) = -x^2, & 0\leq x\leq 1 \end{cases}$$

using separation of variables.
Hint: Write $v(x,t) = u(x,t)+1$. Then v solves the heat equation, but with different boundary conditions and initial conditions. Which? You then have to then have to go through the analysis from class, but with the new boundary conditions.

3

Solve the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x\in\mathbb{R}, \quad t>0 \\ u(x,0) = e^{-rx^2} \end{cases}$$

where $r>0$ is a constant.
Hint: We know a formula for the solution. Use Fourier transform.