Repetitions Part I

1. Laplace transform:

Definition: $f(t),\ t \geq 0$

$$F(s) := \int_0^\infty e^{-st} f(t)\,dt =: \mathcal{L}(f)$$

Problem / Goal: Provides a tool to transform ordinary differential equations (ODEs) of the form

$$\begin{cases} y'(t) + a\,y(t) = r(t), & t \geq 0 \\ y(0) = k_0 \end{cases}$$

or

$$\begin{cases} y''(t) + a\,y'(t) + b\,y(t) = r(t), & t \geq 0 \\ y(0) = k_0 \\ y'(0) = k_1 \end{cases}$$

into algebraic equations.

Theory:

- Existence of Laplace transform (growth condition + piecewise continuous)
- Inverse of Laplace transform

Theory:

- Linearity
- 1st shift theorem: $\mathcal{L}(f) = F(s)$ given for $s > k$, then

$$\mathcal{L}(e^{at} f(t)) = F(s-a) \quad s-a > k.$$

$$e^{at} f(t) = \mathcal{L}^{-1}(F(s-a)).$$

Examples: $\mathcal{L}$ of exponentially damped / amplified oscillations

$$\mathcal{L}(e^{at} \cos \omega t)$$

- Laplace transform of derivatives:

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$- \dots - f^{(n-1)}(0).$$

- Allows us to transform ODE into algebraic equations by "translating derivatives into multiplication with powers of s"
- "Minor version":

$$\mathcal{L}(t f(t)) = -F'(s).$$

- Laplace transform of integrals:

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}F(s) = \frac{1}{s}\mathcal{L}(f)$$

$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(f)\right).$$

Examples: Compute $\frac{1}{s}\cdot\frac{1}{(s^2+a^2)}$ $= \mathcal{L}\left(\frac{\sin(at)}{a}\right)$

- General recipe to solve

$$\begin{cases} y'' + ay' + by = r(t) & a,b \text{ constants.} \\ y(0) = K_0 \\ y'(0) = K_1. \end{cases}$$

1. Apply $\mathcal{L}$ to ODE:

$$R(s) := \mathcal{L}(r) = \mathcal{L}(y'' + ay' + by)$$

$$= (s^2+as+b)\,Y(s) - (s+a)\underbrace{y(0)}_{=K_0} - \underbrace{y'(0)}_{=K_1}$$

2. Introduce transfer function $Q(s) := \dfrac{1}{s^2+as+b}$ and rearrange terms

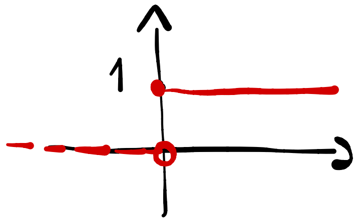
$$Y(s) = \left[(s+a)K_0 + K_1\right]Q(s) + R(s)\,Q(s)$$

3. Compute $\mathcal{L}^{-1}$ of $Y(s)$ to find $y(t)$.

$$y(t) = \mathcal{L}^{-1}(Y).$$

Theory:

- Heaviside function $u(t)$:



- 2nd shift theorem

$$\mathcal{L}\left(f(t-a)\,u(t-a)\right) = e^{-as} F(s).$$

$$f(t-a)\,u(t-a) = \mathcal{L}^{-1}\left(e^{-as} F(s)\right).$$

- Delta function:

  - Formally: $\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases}$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

  - δ is rather a generalized function / a functional

$$f \longmapsto \delta(f) := f(0) = \text{"} \int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt \text{"}.$$

  - Dirac is the derivative of the Heaviside function

$$\delta(t) = u'(t)$$

Theory

- Convolution $h(t) := (f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau.$

- Convolution theorem

$$\mathcal{L}(f*g) = \mathcal{L}(f) \cdot \mathcal{L}(g) = F(s) \cdot G(s)$$

$$f * g = \mathcal{L}^{-1}(F(s) \cdot G(s)).$$

- Application to non-homogeneous ODE,
e.g. compute $\mathcal{L}^{-1}(P(s)\, Q(s))$

Skills:

- Compute Laplace transforms via integration by parts, shift theorems, transformation of integrals and derivatives, convolution theorem

- Solve 1st and 2nd order ODEs, in particular when step functions or δ is involved.

2. Fourier series

Problem:
- Represent or approximate periodic functions via orthogonal systems $\{\phi_n\}_n$.
- Application to PDEs on bounded domains.

Theory:

- Scalar product for complex functions

$$\langle f, g \rangle := \int_a^b f(x)\,\overline{g(x)}\,dx$$

- Norm $\|f\| = \sqrt{\langle f, f \rangle}$

- Orthogonal system $\{\phi_n\}_n$:

$$\langle \phi_n, \phi_m \rangle = 0 \quad \text{if } n \neq m,$$
$$\langle \phi_n, \phi_n \rangle = \|\phi_n\|^2 \neq 0 \quad \forall n$$

- orthonormal system

$$\langle \phi_n, \phi_m \rangle = \delta_{nm} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m. \end{cases}$$

- **Examples:** On $[-\pi, \pi]$: $\{e^{inx}\}_{n\in\mathbb{Z}}$,
  $\{1\} \cup \{\cos nx\}_{n=1}^{\infty} \cup \{\sin nx\}_{n=1}^{\infty}$.

- Application of Gram-Schmidt orthogonalization methods to generate orthogonal / orthonormal systems

- (Abstract) Fourier coefficients $c_n := \langle f, \phi_n \rangle =: c_n(f)$
- Best approximation theorem for general f and finite orthonormal systems $\{\phi_n\}_{n=1}^{N}$:

$$f_N(x) := \sum_{n=1}^{N} c_n(f)\,\phi_n(x)$$

Fourier coefficients

$$g(x) := \sum_{n=1}^{N} d_n\,\phi_n(x)$$

arbitrary coefficients

Then

$$\|f - f_N\| \leq \|f - g\|$$

- Bessel's inequality
- General and trigonometric Fourier series (complex and real)
- Convergence theorem for trigonometric Fourier series
- Transformation between complex and real Fourier series
- trigonometric Fourier series on intervals $[a,b]$.
- Even and odd extensions of f and their Fourier series ⟹ important for PDEs later.
- Parseval's identity & applications (computation of the value of some series

Example: $x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \Rightarrow \sum \frac{1}{n^2} = \frac{\pi^2}{6}$

Skills:

- Compute real and complex Fourier series to given function
- Be able to orthogonalize given set of functions
- Compute best approximations

3. Fourier transform

Problem:
- "Similar" analysis of non-periodic functions as we did for periodic functions
- Application to PDEs on unbounded domains

Theory:

Definitions: $f : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$)

$$\mathcal{F}(f)(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx =: \hat{f}(w)$$

$$\mathcal{F}^{-1}(g)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx}\, dw =: \check{g}(x).$$

- linearity
- Fourier inversion theorem

$$f(x) = \mathcal{F}^{-1}(\hat{f})$$

- Convolution $\quad f * g(x) := \int_{-\infty}^{\infty} f(y) g(x-y)\, dy$
- Convolution theorem

$$\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g)$$

- Differentiation $\quad \mathcal{F}(f') = iw\, \mathcal{F}(f).$

Last two theorems important for solving PDEs.

Skills:
- Compute $\mathcal{F}$ and $\mathcal{F}^{-1}$, also by applying convolution and differentiation theorem.

4. Heat equations

Problem / Task: Solve the heat equations

$$\partial_t u(x,t) - c^2 \Delta u(x,t) = f(x,t) \quad x \in \Omega,\ t>0$$

on various domains (bounded and unbounded) Ω subject boundary conditions and initial condition $u(x,0) = u_0(x)$.

- Various boundary conditions possible
  - Dirichlet b.c. $u(x,t) = u_D(x,t) \quad x \in \partial\Omega$
  - Neumann b.c. $c^2 \nabla u \cdot \vec{n} = g_N(x,t) \quad x \in \partial\Omega$
  - Robin b.c. $c^2 \nabla u \cdot \vec{n} + \mathcal{K}(u - u_R) = 0 \quad x \in \partial\Omega.$

- Theory:
  - Heat equation on $[0, L]$:
$$\begin{cases} \partial_t u = c^2 \partial_x^2 u & \\ u(0,t) = u(L,t) = 0 & \text{b.c.} \\ u(x,0) = f(x) & \text{i.c.} \end{cases}$$
    - Separation of variables ansatz
$$u(x,t) = F(x)\,G(t) \quad \text{leads to}$$
$$\Rightarrow \quad \frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} \quad \text{to find a particular solutions}$$
    - Analysis reveals that: $k > 0$
$$F(x) = A \cos\sqrt{k}\,x + B \sin\sqrt{k}\,x$$
    - Dirichlet b.c. $\Rightarrow$ $A = 0$, $k = \left(\frac{n\pi}{L}\right)^2$
    - Solve now for $G(t)$, which for each $k = \left(\frac{n\pi}{L}\right)^2$ gives a solution: $G_n(t) = e^{-\left(\frac{cn\pi}{L}\right)^2 t}$
    - Now superimpose G_n to generate solution for general f.

Theorem

The heat equation problem

$$\begin{cases} \partial_t u - c^2 \partial_{xx} u = 0 \\ u(0,t) = u(L,t) = 0 \quad \text{Homogeneous Dirichlet conditio} \\ u(x,0) = f(x) \quad \text{Inhomogeneous initial conditions} \end{cases}$$

is solved by

$$\boxed{\begin{gathered} u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right) \\ A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) . \end{gathered}}$$

with

- Heat equation on infinite rod

  - Use Fourier transform to transform PDE with respect to x => leads to ODE

$$\partial_t \hat{u}(w,t) = -c^2 w^2 \hat{u}(w,t)$$

  - Solving ODE gives

$$\hat{u}(w,0) = A(w) e^{-c^2 w^2 t}$$

  - Initial condition

$$\hat{u}(w,t) = A(w) = \hat{f}(w).$$

  - Convolution theorem for $\mathcal{F}$ gives

$$u(x,t) = \frac{1}{\sqrt{2\pi}} f * \mathcal{F}^{-1}\left(e^{-c^2 w^2 t}\right)$$

$$= f * \left(\frac{1}{c\sqrt{4\pi t}} e^{-\frac{x^2}{4c^2 t}}\right)$$

  - Define Heat kernel by

$$g_t(x) = g(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

    then $u(x,t) = (f * g_{c^2 t})(x).$

  - Heat kernel is an "approximation of unity", approximates δ-function for $t \to \infty$.

- ## Theorem 2

The heat equation on x-axis with initial condition

$$u(x,0) = f(x)$$

can be solved by

$$\boxed{u(x,t) = (f * g_{c^2t})(x) = \int_{-\infty}^{\infty} f(v) \frac{1}{c\sqrt{4\pi t}} e^{-\frac{(x-v)^2}{4c^2t}}},$$

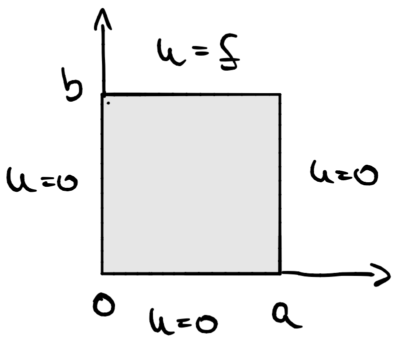
and we have that

$$\lim_{t \to 0} u(x,t) = f(x).$$

- Laplace equations in a rectangular domain

  - Stationary version of heat equations where $\partial_t u = 0 \ \forall t$ (equilibrium).

$$\partial_x^2 u + \partial_y^2 u = 0 \quad + \text{ b.c.}$$



- $\Omega = [0,a] \times [0,b]$
- $u(x,0) = u(0,y) = u(a,y) = 0$
  $u(x,b) = f(x)$

  - Same idea Separation of variables leads to

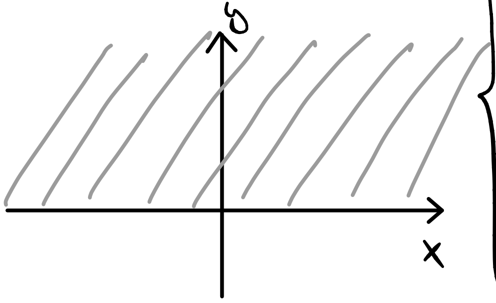
$$\boxed{u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi}{a} x\right)}$$

with $\boxed{A_n = \left(\sinh \frac{n\pi b}{a}\right)^{-1} \cdot \frac{2}{a} \cdot \int_0^a f(x) \sin\left(\frac{n\pi}{a} x\right) dx}$.

• Laplace equations in the half-plane

• $\Omega = \{ (x,y) \in \mathbb{R}^2 : y \geq 0 \}$

• Want to solve



$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0 & 1) \\ \lim\limits_{x \to \pm\infty} u(x,y) = \lim\limits_{y \to \infty} u(x,y) = 0 & 2) \\ u(x,0) = f(x) \text{ with } \lim\limits_{x \to \pm\infty} f(x) = 0. & 3) \end{cases}$$

• Fourier transform technique leads to

$$\boxed{\begin{aligned} u(x,y) &= \frac{1}{\sqrt{2\pi}} f * \left(\sqrt{\frac{2}{\pi}} \frac{y}{y^2 + x^2} \right) \\ &= \int_{-\infty}^{\infty} f(t) \cdot \frac{1}{\pi} \frac{y}{y^2 + (x-t)^2} dt . \end{aligned}}$$

$$= (f * P_y)(x)$$

$P_y(x) = \dfrac{1}{\pi} \dfrac{y}{y^2 + x^2}$ is the Poisson kernel for the upper half space.

Skills:

- Solve heat and Laplace equation on bounded and unbounded domains using separation of variables, Fourier series and Fourier transform techniques
- Understand how to incorporate various boundary conditions and initial conditions in solution procedures.

5. Wave equations

Problem / Task:

$$\partial_t^2 u(x,t) = c^2 \, \partial_x^2 u(x,t) \qquad x \in (0,l),\ t > 0.$$

+ Dirichlet b.c. $u(0,t) = u(l,t) = 0.$

- We need also some initial condition:

i.c. I $\quad u(x,0) = f(x)$ (initial displacement).

i.c. II $\quad \partial_t u(x,0) = g(x)$ (initial velocity).

Theory / techniques:

- Techniques similar to heat equations
- Solution on interval via Separation of variables

Theorem 1

The wave equation of the form $\boxed{1}$ has the solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{cn\pi}{l} t + B_n \sin \frac{cn\pi}{l} t \right) \sin \frac{n\pi}{l} x$$

with

$$\boxed{\begin{aligned} A_n &= \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx, \\ B_n &= \frac{2}{cn\pi} \int_0^l g(x) \sin \frac{n\pi}{l} x \, dx. \end{aligned}}$$

- **Solution on x-axis**

- d'Alembert's ansatz:

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

leads to

Theorem 2

The problem

$$\begin{cases} \partial_t^2 u(x,t) = c^2 \partial_x u(x,t) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & \text{i.c. I} \\ \partial_t u(x,0) = g(x) & \text{i.c. II} \end{cases}$$

has the solution

$$\boxed{u(x,t) = \frac{1}{2}\left(f(x+ct) - f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.}$$

- **Skills:** Similar as for the heat equation, being able to solve wave equation on bounded and unbound intervals via Separation of variables, Fourier series or d'Alembert.