Norwegian University of Science and Technology
Department for Mathematical Science

TMA4125 Calculus 4N
Spring 2020

Exercise set 8
Nonlinear equations

All necessary functions in Python can be found in the Jupyter notebook `NonLinear-Equations.ipynb`.

Mandatory exercises

1 Given the equation $f(x) = e^x + x^2 - x - 4 = 0$.

a) Show using theory that the function f has a unique root r in the interval $[1, 2]$. Additionally, plot the function f is the interval $[1, 2]$ and find an approximation of r by looking at the graph.
Use Newton's method to find r, and choose an initial value from the graph above. Do a couple of iterations of the algorithm on paper. Find an exact solution by using the function **`newton`** in the Jupyter notebook.

b) The equation $f(x) = 0$ can be written on the form $x = g(x)$ with for example

$$\begin{aligned} &i) && g(x) = \ln(4 + x - x^2) \\ &ii) && g(x) = \sqrt{-e^x + x + 4} \\ &iii) && g(x) = e^x + x^2 - 4 \end{aligned}$$

For each of these functions we can make a fixed.point scheme $x_{n+1} = g(x_n)$.

Test the three schemes numerically by using the function **`fixpunkt`**, and see which scheme converges. Use $x_0 = 1.5$, but we encourage you to experiment with different initial values. Use the statement on fixed-point iterations to explain the result.
Hint: Plot the functions $g(x)$ and $g'(x)$. If necessary, choose a different interval $[a, b]$ containing r.
For those schemes that converges, estimate the number of iterations necessary to get an accuracy of 10^{-6}. Check the answer numerically. (The answer depends on the interval you choose, so there is no definite answer.)

2 **a)** Let $g(x)$ be a continuous function with continuous derivatives on (a, b) and suppose that it has an inverse $g^{-1}(x)$. Show that if $r \in (a, b)$ is a fixed-point of $g(x)$, then r is also a fixed-point of $g^{-1}(x)$.

b) Let $r \in (a, b)$ be a fixed-point of $g(x)$. Show that if $|g'(r)| > 1$, then $|(g^{-1})'(r)| < 1$.

Using the convergence theorem of fixed-point iteration, we now know that if the algorithm does not converge for $g(x)$, then it will converge for $g^{-1}(x)$ if one takes x_0 sufficiently close to r.

c) With this in mind, use fixed-point iteration to find an approximation to the solution r of the equation $x = \arccos(x)$.

3 Use Newton's method to find $\sqrt[3]{7}$.

a) Write down the iteration scheme, and perform the first few iterations by hand, using $x_0 = 1$.

b) Confirm your results by using the function **`newton`** in the Jupyter notebook.

Recommended exercises

4 **a)** Compute 3 iterations by hand of Newton's method to approximate a solution to the system of equations

$$x^2 + y^2 = 4$$
$$xy = 1$$

starting with $x_0 = 2$, $y_0 = 0$.

b) Find an approximation to the solution by using the function **`newton_sys`**. You can also use this to confirm that your hand calculations were correct.

c) Use **`newton_sys`** to solve the slightly perturbed system

$$x^2 + y^2 = 2$$
$$xy = 1$$

with the same initial values. How does the method behave now? Explain the results.